Advanced Mathematical Models & Applications
Vol.8, No.3, 2023, pp.464-484

GLOBAL STABILITY ANALYSIS OF TRANSMISSION DYNAMICS OF A MODEL OF MALARIA WITH FUZZY MOSQUITO POPULATIONS

Yves Tinda Mangongo[1*], **Joseph-Désiré Kyemba Bukweli**[1], **Justin Dupar Busili Kampempe**[1], **Justin Manango Wazute Munganga**[2]

[1]Department of Mathematics and Computer Science, University of Kinshasa, Kinshasa, DRCongo
[2]Department of Mathematical Sciences, University of South Africa, Pretoria, South Africa

Abstract. In this article, we analyze a model of malaria dynamics transmission in a fuzzy environment by assuming that the mosquito populations are fuzzy variables. From the fuzzy system of differential equations, we derive its deterministic counterpart using level set α of a fuzzy number, with $\alpha \in [0,1]$. We prove that in the absence of malaria, the human and mosquito populations belong to the level set α of the fuzzy susceptible humans $\tilde{S}_h$ and the fuzzy susceptible mosquitoes $\tilde{S}_m$ respectively. We prove that the disease-free equilibrium is globally asymptotically stable if the fuzzy basic reproduction number $\mathcal{R}_0^\alpha$ is less than 1 for all level $\alpha \in]0,1]$. When $\mathcal{R}_0^\alpha > 1$, the disease-free equilibrium is unstable. Several possibilities for different levels $\alpha \in]0,1]$ are analyzed. To eliminate malaria, analysis must be done for values of α very closer to unity for all fuzzy variables of the model.

Keywords: Malaria, Ignorant infected humans, Fuzzy mosquito populations, Deterministic counterpart, Fuzzy basic reproduction number, Fuzzy differential equations.
AMS Subject Classification: 97M10.
Corresponding author: Yves Tinda Mangongo, Department of Mathematics and Computer Science, Kinshasa, DRCongo, Tel.: +243852184768, e-mail: *yves.mangongo@unikin.ac.cd*
Received: 23 June 2023; Revised: 18 July 2023; Accepted: 3 September 2023; Published: 20 December 2023.

1 Introduction

Malaria, one of the oldest diseases, remains a major health problem worldwide, especially in the low-income countries (O.M.S., 2022; Olaniyi et al., 2022). In 2021, nearly half of the world's population was at risk of malaria. The estimated number of malaria deaths stood at 619 000 (W.H.O.). Malaria is caused by a parasite named *"Plasmodium"*, the parasites are transmitted through the bites of an infectious female *Anopheles* (W.H.O, 2010; PNLP, Mars 2018). Among the *"Plasmodium"* that wreak most havocs on human population are *"Plasmodium"* falciparum and vivax (W.H.O.).

Mathematical modeling provides a powerful tool to describe and to analyze the spread process of malaria and its complex mechanisms of transmission. In deterministic (standard) approach, many studies have been done: Djidjou-Demasse et al. (2020), Olaniyi et al. (2020), Olaniyi et al. (2022), Atangana & Qureshi (2020), Ndamuzi & Gahungu (2021), Herdicho et al. (2021), Mangongo et al. (2022), Olaniyi et al. (2022), Keno et al. (2022) and references therein, since Ronald Ross in 1910. Some of them are interested in seasonal mosquito life-history traits and

How to cite (APA): Y.T. Mangongo, J.-D.K. Bukweli, J.D.B. Kampempe, & J.M.W. Munganga (2023). Global stability analysis of transmission dynamics of a model of Malaria with fuzzy mosquito populations. *Advanced Mathematical Models & Applications*, *8*(3), 464-484.

optimal control of the disease. For more details, see Djidjou-Demasse et al. (2020), Olaniyi et al. (2020), Olaniyi et al. (2022), Atangana & Qureshi (2020), Ndamuzi & Gahungu (2021), Herdicho et al. (2021) and references therein. Some of them used caputo derivative to design models (Park et al., 2020). Mangongo et al. (2022) recently designed and analyzed a mathematical model for malaria transmission dynamics, in which they introduced the concept of ignorant infected humans. They proved that ignorant infected humans are those who influence very significantly the spread of malaria in the population. Olaniyi et al. (2022) analyzed a social hierarchy-structured model for malaria transmission dynamics. They assumed that the infection of malaria depends on the social classes to which the individuals belong. An optimal control analysis of malaria epidemic in the presence of temperature variability was proposed by Keno et al. (2022). They analyzed the impact of temperature variability on the transmission dynamics of malaria. They used the Pontraygin's maximum principle to obtain the necessary condition for the optimal control problem.

To incorporate the high degree of subjectivity and imprecision in modeling diseases, some studies used fuzzy set theory introduced by Zadeh (1965), we can cite: De Barros et al. (2003), De Barros et al. (2000), Alamin et al. (2020), Mondal et al. (2015), Verma et al. (2018), Singh et al. (2023). Many other real world situations used fuzzy set theory to model problems such as in artificial intelligence, mathematical programming, etc. (Gazi et al., 2023; Momena et al., 2023). In the same vein Dayan et al. (2022a,b) and Allehiany et al. (2022) have been done studies in fuzzy modeling of infectious diseases. For mosquito-borne diseases, we have the work of Bhuju et al. (2020), who used fuzzy set theory to design and analyze a model of dengue disease in Nepal. They assumed that transmission and recovered rates are fuzzy numbers depending on the viral load. Mangongo et al. (2021a) proposed a fuzzy analysis of the transmission dynamics of malaria by assuming that transmission and recovered rates are fuzzy numbers depending on the parasites density. They introduced the fuzzy basic reproduction number, the fuzzy analysis of equilibria and fuzzy global asymptotic stability analysis of the disease free equilibrium. Dayan et al. (2022c) studied the dynamics of a fuzzy epidemic model of a mosquito-borne disease. In their paper, they assumed that the chance of occurrence of dengue infection, the recovered and mortality rates of humans populations are fuzzy numbers. Singh et al. (2023) discussed a mathematical model focused on severely infected people of malaria transmission in both crisp and fuzzy environments.

It is well-know that, one of the malaria's problem is the imprecision or vagueness in the size of mosquito populations, which is not well-known in reality. To incorporate this vagueness in a model of malaria dynamics transmission, we assume that the mosquito populations are fuzzy variables. We will see the implication of a such assumption in biological context of malaria.

The rest of paper is organized as follows: in section 2, we formulate the model, in which we assume that, the mosquito populations are triangular fuzzy variables, and we derived the deterministic counterpart of the fuzzy initial model. In section 3, the deterministic counterpart of the fuzzy initial model is analytically analyzed, then we prove its wellposedness, and analyze the global asymptotic stability of the disease free equilibrium and the endemic equilibrium using the fuzzy basic reproduction number. In section 4, some numerical analysis are done. We give concluding remarks in section 5.

2 Formulation of the model

In this section, we analyze a model of malaria dynamics transmission in which we suppose that the mosquito populations are fuzzy variables. This assumption is justified by the fact that in the modeling of malaria, it's more difficult to estimate the exact number of mosquito populations in a given environment for a certain period, knowing the short life-span of mosquito populations. The human population is subdivided into five compartments: susceptible S_h, exposed E_h, infectious I_h, recovered R_h and ignorant infected M_h humans. The ignorant infected humans are those

who still have *Plasmodium* parasites after a certain period of treatment without knowing it and they can relapse to the malaria at a rate φ if their immune systems decrease. For the mosquito populations, we use the classical SEI scheme by incorporating the fuzziness in each state variable. The latent period of humans is $\frac{1}{\sigma}$, where σ is the latent rate of humans. The infectious humans recover at a rate γ and they can die due to malaria at a rate δ. The recovered humans loss their immunity and become susceptible again at a rate η. We denote μ the natural death rate of human populations. We assume that the susceptible human population increases by a constant recruitment Λ_h. The incubation period of mosquitoes is equal to $\frac{1}{\sigma_m}$, where σ_m is the incubation rate of mosquitoes. We denote μ_m the natural death and birth rates of mosquitoes. The flow diagram describing the fuzzy model is given on Figure (1). We assume that all model parameters are crisp numbers, and they are summarized in Table (1).

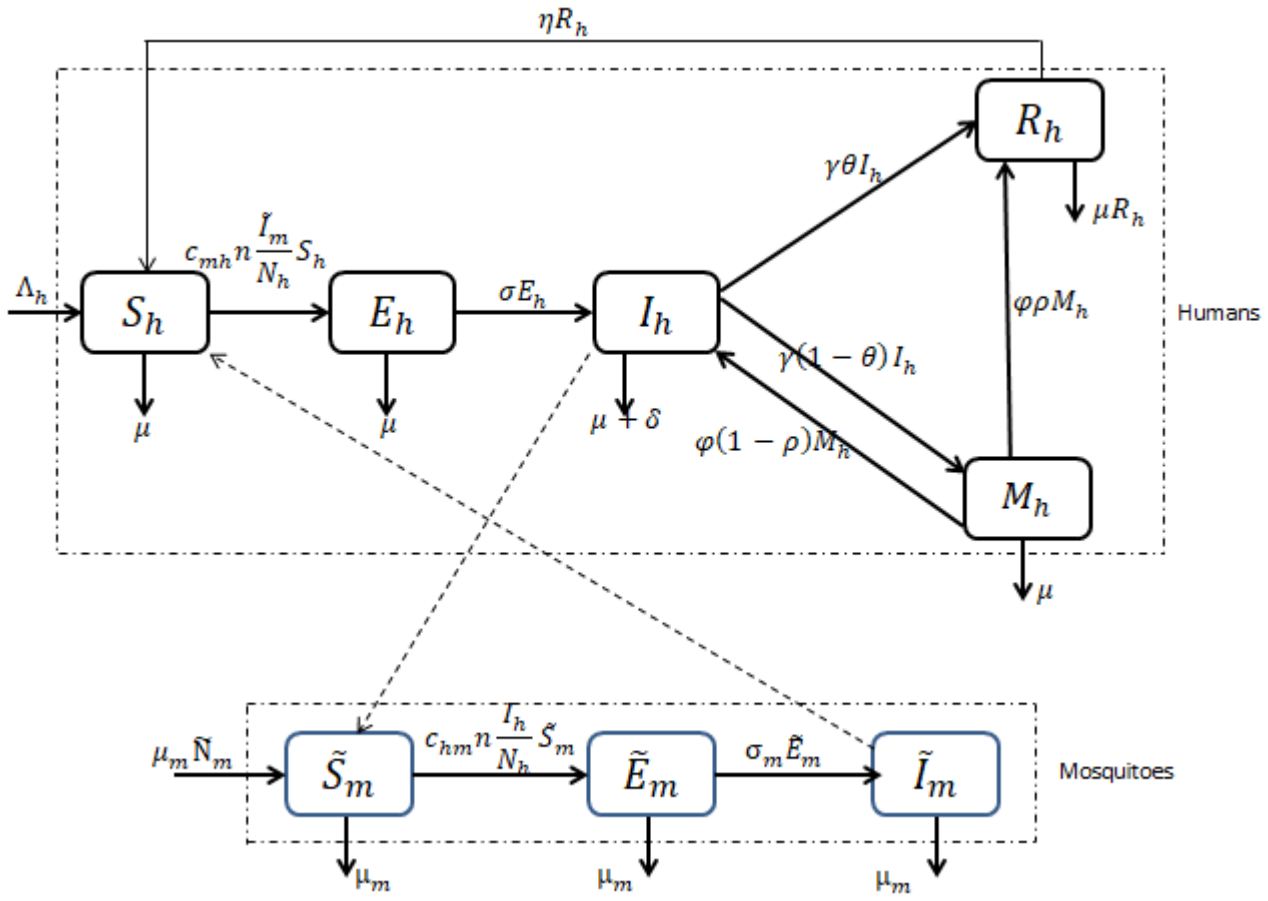

Figure 1: Fuzzy transfer diagram of malaria transmission

The mosquito populations are described as follow: $\tilde{S}_m, \tilde{E}_m$ and $\tilde{I}_m$ are the fuzzy susceptible, the fuzzy exposed and the fuzzy infectious mosquito populations respectively. We assume that all the fuzzy state variables are triangular fuzzy variables. At each time the human population is given by

$$N_h(t) = S_h(t) + E_h(t) + I_h(t) + R_h(t) + M_h(t), \tag{1}$$

and the fuzzy mosquito population is given by

$$\tilde{N}_m(t) = \tilde{S}_m(t) + \tilde{E}_m(t) + \tilde{I}_m = (N_{m1}, N_{m2}, N_{m3}), \tag{2}$$

where " ˜ " represents the fuzziness in a given variable. The concepts about fuzzy sets theory can be found in Zadeh (1965); Mangongo et al. (2021b); Fard (2009); Bede et al. (2013); Buckley & Feuring (2000); Seikkala (1987).

Definition 1. *The fuzzy law of mass actions of mosquitoes to humans and for humans to mosquitoes are given respectively by:*

$$\tilde{\lambda}_h = c_{mh} n \frac{\tilde{I}_m}{N_h} S_h, \quad and \quad \tilde{\lambda}_m = c_{hm} n \frac{I_h}{N_h} \tilde{S}_m. \tag{3}$$

From the transfer diagram of Figure 1, we derive the system (4)-(11) of fuzzy differential equations with crisp constants.

Table 1: Parameters, meanings, baseline values, dimension and references

Param.	**Meaning of parameters**	**Values**	Dimensions	**Ref.**
Λ_h	constant recruitment of susceptible humans	1	*hum./day*	assumed
σ	latent rate of humans	0.0588	*hum./day*	(Olaniyi et al., 2020)
γ	recovery rate of humans	0.071	*hum./day*	(Mangongo et al., 2022)
θ	proportion of infectious humans who recover	0.2	-	(Mangongo et al., 2022)
$1-\theta$	proportion of human who still have *Plasmodium* parasites	0.8	-	-
μ	natural death rate of humans	421×10^{-7}	*hum./day*	(Mangongo et al., 2022)
δ	lethality rate	1285×10^{-7}	*hum./day*	(Mangongo et al., 2022)
η	progression rate of R_h into S_h	0.02	*hum./day*	(Mangongo et al., 2022)
ρ	proportion of ignorant infected humans who recover	0.005	-	(Mangongo et al., 2022)
$1-\rho$	proportion of ignorant infected humans who relapse	0.995	-	-
φ	relapse rate	0.1	*hum./day*	(Mangongo et al., 2022)
n	the average number of mosquito bites	4	*bites/day*	(Zongo, 2009)
c_{mh}	probability that a bite by an infectious mosquito to a susceptible human leads to an infection of this susceptible human	0.07	-	(Zongo, 2009)
c_{hm}	probability that a bite by a susceptible mosquito to an infectious human leads to an infection of this susceptible mosquito	0.07	-	(Zongo, 2009)
μ_m	natural death rate of mosquitoes	435×10^{-4}	*mosq./day*	(Ndoen et al., 2012)
σ_m	incubation rate of mosquitoes	71×10^{-3}	*mosq./day*	(Adugna et al., 2022)

$$\dot{S}_h = \Lambda_h + \eta R_h - \left(c_{mh} n \frac{\tilde{I}_m}{N_h} + \mu \right) S_h \tag{4}$$

$$\dot{E}_h = c_{mh} n \frac{\tilde{I}_m}{N_h} S_h - (\sigma + \mu) E_h \tag{5}$$

$$\dot{I}_h = \sigma E_h + \varphi(1-\rho) M_h - (\gamma + \mu + \delta) I_h \tag{6}$$

$$\dot{R}_h = \gamma\theta I_h + \varphi\rho M_h - (\eta + \mu) R_h \tag{7}$$

$$\dot{M}_h = \gamma(1-\theta) I_h - (\varphi + \mu) M_h \tag{8}$$

$$\dot{\tilde{S}}_m = \mu_m \tilde{N}_m - \left(c_{hm} n \frac{I_h}{N_h} + \mu_m \right) \tilde{S}_m \tag{9}$$

$$\dot{\tilde{E}}_m = c_{hm} n \frac{I_h}{N_h} \tilde{S}_m - (\sigma_m + \mu_m) \tilde{E}_m \tag{10}$$

$$\dot{\tilde{I}}_m = \sigma_m \tilde{E}_m - \mu_m \tilde{I}_m \tag{11}$$

The system (4)-(11) is solved under the following non-negatives initial conditions:

$$\begin{aligned} &S_h(0) \geq 0, E_h(0) \geq 0, I_h(0) \geq 0, R_h(0) \geq 0, \\ &M_h(0) \geq 0, \tilde{S}_m(0) \geq 0, \tilde{E}_m(0) \geq 0, \tilde{I}_m(0) \geq 0. \end{aligned} \tag{12}$$

Notations 1. *The fuzzification of variables S_h, E_h, I_h, R_h and M_h into triangular fuzzy variables give: $\tilde{S}_h = (S_{h1}, S_{h2}, S_{h3}), \tilde{E}_h = (E_{h1}, E_{h2}, E_{h3}), \tilde{I}_h = (I_{h1}, I_{h2}, I_{h3}), \tilde{R}_h = (R_{h1}, R_{h2}, R_{h3})$ and $\tilde{M}_h = (M_{h1}, M_{h2}, M_{h3})$ respectively. The level sets α of $\tilde{S}_m, \tilde{E}_m$ and $\tilde{I}_m$ are given by $\tilde{S}_m^\alpha = [S_{m1}^\alpha, S_{m2}^\alpha]$, $\tilde{E}_m^\alpha = [E_{m1}^\alpha, E_{m2}^\alpha]$ and $\tilde{I}_m^\alpha = [I_{m1}^\alpha, I_{m2}^\alpha]$ respectively and the one of $\tilde{S}_h, \tilde{E}_h, \tilde{I}_h, \tilde{R}_h$ and $\tilde{M}_h$ are given by $\tilde{S}_h^\alpha = [S_{h1}^\alpha, S_{h2}^\alpha]$, $\tilde{E}_h^\alpha = [E_{h1}^\alpha, E_{h2}^\alpha], \tilde{I}_h^\alpha = [I_{h1}^\alpha, I_{h2}^\alpha], \tilde{R}_h^\alpha = [R_{h1}^\alpha, R_{h2}^\alpha]$ and $\tilde{M}_h^\alpha = [M_{h1}^\alpha, M_{h2}^\alpha]$ respectively. The fuzzification of the constant recruitment Λ_h into a triangular fuzzy number is $\tilde{\Lambda}_h = (\Lambda_{h1}, \Lambda_{h2}, \Lambda_{h3})$ and its level set α is given by $\tilde{\Lambda}_h^\alpha = [\Lambda_{h1}^\alpha, \Lambda_{h2}^\alpha]$.*

Theorem 1 gives the deterministic counterpart of system (4)-(11).

Theorem 1. *The deterministic counterpart of fuzzy differential equations (4)-(11) is given by the system (13)-(28) of non linear differential equations below:*

$$\dot{S}_{h1}^\alpha = \Lambda_{h1}^\alpha + \eta R_{h1}^\alpha - \alpha \left(c_{mh} n \frac{I_{m1}^\alpha}{N_h^\alpha} + \mu \right) S_{h1}^\alpha \tag{13}$$

$$\dot{S}_{h2}^\alpha = \Lambda_{h2}^\alpha + \eta R_{h2}^\alpha - \alpha \left(c_{mh} n \frac{I_{m2}^\alpha}{N_h^\alpha} + \mu \right) S_{h2}^\alpha \tag{14}$$

$$\dot{E}_{h1}^\alpha = c_{mh} n \frac{I_{m1}^\alpha}{N_h^\alpha} S_{h1}^\alpha - (\sigma + \mu) E_{h1}^\alpha \tag{15}$$

$$\dot{E}_{h2}^\alpha = c_{mh} n \frac{I_{m2}^\alpha}{N_h^\alpha} S_{h2}^\alpha - (\sigma + \mu) E_{h2}^\alpha \tag{16}$$

$$\dot{I}_{h1}^\alpha = \sigma E_{h1}^\alpha + \varphi(1-\rho) M_{h1}^\alpha - (\gamma + \mu + \delta) I_{h1}^\alpha \tag{17}$$

$$\dot{I}_{h2}^\alpha = \sigma E_{h2}^\alpha + \varphi(1-\rho) M_{h2}^\alpha - (\gamma + \mu + \delta) I_{h2}^\alpha \tag{18}$$

$$\dot{R}_{h1}^\alpha = \gamma\theta I_{h1}^\alpha + \varphi\rho M_{h1}^\alpha - (\eta + \mu) R_{h1}^\alpha \tag{19}$$

$$\dot{R}_{h2}^\alpha = \gamma\theta I_{h2}^\alpha + \varphi\rho M_{h2}^\alpha - (\eta + \mu) R_{h2}^\alpha \tag{20}$$

$$\dot{M}_{h1}^\alpha = \gamma(1-\theta) I_{h1}^\alpha - (\varphi + \mu) M_{h1}^\alpha \tag{21}$$

$$\dot{M}_{h2}^\alpha = \gamma(1-\theta) I_{h2}^\alpha - (\varphi + \mu) M_{h2}^\alpha \tag{22}$$

$$\dot{S}_{m1}^\alpha = (1-\alpha)\mu_m N_{m1} - \left(c_{hm} n \frac{I_{h1}^\alpha}{N_h^\alpha} + \mu_m \right) S_{m1}^\alpha + \alpha\mu_m N_{m2} \tag{23}$$

$$\dot{S}_{m2}^\alpha = (1-\alpha)\mu_m N_{m3} - \left(c_{hm} n \frac{I_{h2}^\alpha}{N_h^\alpha} + \mu_m \right) S_{m2}^\alpha + \alpha\mu_m N_{m2} \tag{24}$$

$$\dot{E}_{m1}^\alpha = c_{hm} n \frac{I_{h1}^\alpha}{N_h^\alpha} S_{m1}^\alpha - (\sigma_m + \mu_m) E_{m1}^\alpha \tag{25}$$

$$\dot{E}_{m2}^\alpha = c_{hm} n \frac{I_{h2}^\alpha}{N_h^\alpha} S_{m2}^\alpha - (\sigma_m + \mu_m) E_{m2}^\alpha \tag{26}$$

$$\dot{I}_{m1}^\alpha = \sigma_m E_{m1}^\alpha - \mu_m I_{m1}^\alpha \tag{27}$$

$$\dot{I}_{m2}^\alpha = \sigma_m E_{m2}^\alpha - \mu_m I_{m2}^\alpha \tag{28}$$

Proof. Equation (11) can be written as:

$$\dot{\tilde{I}}_m = (\sigma_m E_{m1} - \mu_m I_{m1}, \sigma_m E_{m2} - \mu_m I_{m2}, \sigma_m E_{m3} - \mu_m I_{m3}). \tag{29}$$

Let $f(t) = (\sigma_m E_{m1} - \mu_m I_{m1}, \sigma_m E_{m2} - \mu_m I_{m2}, \sigma_m E_{m3} - \mu_m I_{m3}) \in \mathcal{F}(\mathbb{R})$ and $t \in I \subseteq \mathbb{R}$. The set $\mathcal{F}(\mathbb{R})$ denotes a set of fuzzy numbers. If we assume that the function f is Hukuhara

differentiable, and if the functions $\sigma_m E_{m1} - \mu_m I_{m1}, \sigma_m E_{m2} - \mu_m I_{m2}$ and $\sigma_m E_{m3} - \mu_m I_{m3}$ are reals differentiable functions, then the function f can be written after in the form:

$$f(t) = \begin{cases} \dfrac{t - \sigma_m E_{m1} + \mu_m I_{m1}}{\sigma_m(E_{m2} - E_{m1}) + \mu_m(I_{m1} - I_{m2})} & if \quad \sigma_m E_{m1} - \mu_m I_{m1} < t \leq \sigma_m E_{m2} - \mu_m I_{m2} \\ \dfrac{\sigma_m E_{m3} - \mu_m I_{m3} - t}{\sigma_m(E_{m3} - E_{m2}) + \mu_m(I_{m2} - I_{m3})} & if \quad \sigma_m E_{m2} - \mu_m I_{m2} < t < \sigma_m E_{m3} - \mu_m I_{m3} \\ 0 & elsewhere \end{cases},$$

Therefore for all $\alpha \in [0,1]$, and taking into account the level sets α of I_m and E_m, the level set α of f can be written as:

$$[\dot{\tilde{I}}_m]^\alpha \equiv f_\alpha(t) = [f_\alpha^-(t), f_\alpha^+(t)], \tag{30}$$

where

$$f_\alpha^-(t) = \sigma_m E_{m1}^\alpha - \mu_m I_{m1}^\alpha \ \ and \ \ f_\alpha^+(t) = \sigma_m E_{m2}^\alpha - \mu_m I_{m2}^\alpha.$$

We can do the same for $\dot{\tilde{S}}_m$ and $\dot{\tilde{E}}_m$.

The fuzzification of $\dot{S}_h$ gives:

$$\begin{aligned} \dot{S}_h &= \left(\Lambda_{h1}^\alpha + \eta R_{h1} - \left(c_{mh} n \frac{I_{m1}}{N_h^\alpha} + \mu\right) S_{h1}, \Lambda_{h2}^\alpha + \eta R_{h2} - \left(c_{mh} n \frac{I_{m2}}{N_h^\alpha} + \mu\right) S_{h2}, \Lambda_{h3}^\alpha + \eta R_{h3}\right. \\ &- \left.\left(c_{mh} n \frac{I_{m3}}{N_h^\alpha} + \mu\right) S_{h3}\right). \end{aligned} \tag{31}$$

Let $p(t) = \left(\Lambda_{h1}^\alpha + \eta R_{h1} - \left(c_{mh} n \frac{I_{m1}}{N_h^\alpha} + \mu\right) S_{h1}, \Lambda_{h2}^\alpha + \eta R_{h2} - \left(c_{mh} n \frac{I_{m2}}{N_h^\alpha} + \mu\right) S_{h2}, \Lambda_{h3}^\alpha + \eta R_{h3}\right.$ $\left. - \left(c_{mh} n \frac{I_{m3}}{N_h^\alpha} + \mu\right) S_{h3}\right) \in \mathcal{F}(\mathbb{R})$ and $t \in I \subseteq \mathbb{R}$. If we assume that function p is Hukuhara differentiable and if the functions $\Lambda_{h1}^\alpha + \eta R_{h1} - \left(c_{mh} n \frac{I_{m1}}{N_h^\alpha} + \mu\right) S_{h1}, \Lambda_{h2}^\alpha + \eta R_{h2} - \left(c_{mh} n \frac{I_{m2}}{N_h^\alpha} + \mu\right) S_{h2}$ and $\Lambda_{h3}^\alpha + \eta R_{h3} - \left(c_{mh} n \frac{I_{m3}}{N_h^\alpha} + \mu\right) S_{h3}$ are reals differentiable functions, then for all $\alpha \in [0,1]$, and taking into account the level sets α of S_h, I_m and R_h, the level set α of p can be written as:

$$[\dot{S}_h]^\alpha \equiv p_\alpha(t) = [p_\alpha^-(t), p_\alpha^+(t)], \tag{32}$$

where

$$p_\alpha^-(t) = \Lambda_{h1}^\alpha + \eta R_{h1}^\alpha - \alpha\left(c_{mh} n \frac{I_{m1}^\alpha}{N_h^\alpha} + \mu\right) S_{h1}^\alpha \ \ and \ \ p_\alpha^+(t) = \Lambda_{h2}^\alpha + \eta R_{h2}^\alpha - \alpha\left(c_{mh} n \frac{I_{m2}^\alpha}{N_h^\alpha} + \mu\right) S_{h2}^\alpha.$$

We can do the same for $\dot{E}_h, \dot{I}_h, \dot{R}_h$ and $\dot{M}_h$.

Taking the lower and upper bounds of each level set, we obtain the system (13)-(28), which is the deterministic counterpart of the fuzzy system (4)-(11). □

Remark 1. *The system (13)-(28) gives the lower and upper bounds for each fuzzy differential equation of system (4)-(11). Each solution, bounded between the lower and upper bounds of each level set α, is a candidate solution of the corresponding fuzzy differential equation.*

Notations 2. *The system (13)-(28) is solved under the initial conditions:*

$$S_{h1}^\alpha \geq 0, S_{h2}^\alpha \geq 0, E_{h1}^\alpha \geq 0, E_{h2}^\alpha \geq 0, I_{h1}^\alpha \geq 0, I_{h2}^\alpha \geq 0, R_{h1}^\alpha \geq 0, R_{h2}^\alpha \geq 0, \tag{33}$$

$$M_{h1}^\alpha \geq 0, M_{h2}^\alpha \geq 0, S_{m1}^\alpha \geq 0, S_{m2}^\alpha \geq 0, E_{m1}^\alpha \geq 0, E_{m2}^\alpha \geq 0, I_{m1}^\alpha \geq 0, I_{m2}^\alpha \geq 0. \tag{34}$$

The lower bounds of the total human and mosquito populations of the model (13)-(28) are given respectively by:

$$N_{h1}^{\alpha} = S_{h1}^{\alpha} + E_{h1}^{\alpha} + I_{h1}^{\alpha} + R_{h1}^{\alpha} + M_{h1}^{\alpha} \ \ and \ \ N_{m1}^{\alpha} = S_{m1}^{\alpha} + E_{m1}^{\alpha} + I_{m1}^{\alpha}. \tag{35}$$

The upper bounds of the total human and mosquito populations of the model (13)-(28) are given respectively by:

$$N_{h2}^{\alpha} = S_{h2}^{\alpha} + +E_{h2}^{\alpha} + I_{h2}^{\alpha} + R_{h2}^{\alpha} + M_{h2}^{\alpha} \ \ and \ \ N_{m2}^{\alpha} = S_{m2}^{\alpha} + E_{m2}^{\alpha} + I_{m2}^{\alpha}. \tag{36}$$

3 Mathematical analysis of the model

In this section, we do the analytical analysis of the model (13)-(28). The wellposedness, equilibrium of the model and computation of the fuzzy basic reproduction number are presented. Finally, global stabilities analysis of the disease free and endemic equilibrium are provided.

3.1 Wellposedness of the model

Let given $Y(t) = (S_{h1}^{\alpha}, S_{h2}^{\alpha}, E_{h1}^{\alpha}, E_{h2}^{\alpha}, I_{h1}^{\alpha}, I_{h2}^{\alpha}, R_{h1}^{\alpha}, R_{h2}^{\alpha}, M_{h1}^{\alpha}, M_{h2}^{\alpha}, S_{m1}^{\alpha}, S_{m2}^{\alpha}, E_{m1}^{\alpha}, E_{m2}^{\alpha}, I_{m1}^{\alpha}, I_{m2}^{\alpha})$ and $f : \Omega \subseteq \mathbb{R}_{+}^{16} \to \subseteq \mathbb{R}_{+}^{16}$; $Y(t) \mapsto f(Y(t)) \equiv \dot{Y}(t)$, where the feasible set Ω is defined by:

$$\Omega = \Big\{(S_{h1}^{\alpha}, S_{h2}^{\alpha}, E_{h1}^{\alpha}, E_{h2}^{\alpha}, I_{h1}^{\alpha}, I_{h2}^{\alpha}, R_{h1}^{\alpha}, R_{h2}^{\alpha}, M_{h1}^{\alpha}, M_{h1}^{\alpha}, S_{m1}^{\alpha}, S_{m2}^{\alpha}, E_{m1}^{\alpha}, E_{m2}^{\alpha}, I_{m1}^{\alpha}, I_{m2}^{\alpha}) \in \mathbb{R}_{+}^{16} : N_{h1}^{\alpha}(t) \leq \frac{\Lambda_{h1}^{\alpha}}{\mu}, N_{h2}^{\alpha}(t) \leq \frac{\Lambda_{h2}^{\alpha}}{\mu}, N_{m1}^{\alpha}(t) \leq N_{m1,0}^{\alpha} \ and \ N_{m2}^{\alpha}(t) \leq N_{m2,0}^{\alpha}\Big\}, \tag{37}$$

with $N_{m1,0}^{\alpha}$ and $N_{m2,0}^{\alpha}$ the initial total lower and upper bounds of mosquito populations respectively. The function $f(Y(t))$ is defined by
$f(Y(t)) = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16})$, where:

$$\left\{\begin{array}{rcl}
f_1(Y(t)) &=& \Lambda_{h1}^{\alpha} + \eta R_{h1}^{\alpha} - \alpha\left(c_{mh} n \frac{I_{m1}^{\alpha}}{N_h^{\alpha}} + \mu\right) S_{h1}^{\alpha} \\
f_2(Y(t)) &=& \Lambda_{h2}^{\alpha} + \eta R_{h2}^{\alpha} - \alpha\left(c_{mh} n \frac{I_{m2}^{\alpha}}{N_h^{\alpha}} + \mu\right) S_{h2}^{\alpha} \\
f_3(Y(t)) &=& c_{mh} n \frac{I_{m1}^{\alpha}}{N_h^{\alpha}} S_{h1}^{\alpha} - (\sigma + \mu) E_{h1}^{\alpha} \\
f_4(Y(t)) &=& c_{mh} n \frac{I_{m2}^{\alpha}}{N_h^{\alpha}} S_{h2}^{\alpha} - (\sigma + \mu) E_{h2}^{\alpha} \\
f_5(Y(t)) &=& \sigma E_{h1}^{\alpha} + \varphi(1-\rho) M_{h1}^{\alpha} - (\gamma + \mu + \delta) I_{h1}^{\alpha} \\
f_6(Y(t)) &=& \sigma E_{h2}^{\alpha} + \varphi(1-\rho) M_{h2}^{\alpha} - (\gamma + \mu + \delta) I_{h2}^{\alpha} \\
f_7(Y(t)) &=& \gamma\theta I_{h1}^{\alpha} + \varphi\rho M_{h1}^{\alpha} - (\eta + \mu) R_{h1}^{\alpha} \\
f_8(Y(t)) &=& \gamma\theta I_{h2}^{\alpha} + \varphi\rho M_{h2}^{\alpha} - (\eta + \mu) R_{h2}^{\alpha} \\
f_9(Y(t)) &=& \gamma(1-\theta) I_{h1}^{\alpha} - (\varphi + \mu) M_{h1}^{\alpha} \\
f_{10}(Y(t)) &=& \gamma(1-\theta) I_{h2}^{\alpha} - (\varphi + \mu) M_{h2}^{\alpha} \\
f_{11}(Y(t)) &=& (1-\alpha)\mu_m N_{m1} - \left(c_{hm} n \frac{I_{h1}^{\alpha}}{N_h^{\alpha}} + \mu_m\right) S_{m1}^{\alpha} + \alpha\mu_m N_{m2} \\
f_{12}(Y(t)) &=& (1-\alpha)\mu_m N_{m3} - \left(c_{hm} n \frac{I_{h2}^{\alpha}}{N_h^{\alpha}} + \mu_m\right) S_{m2}^{\alpha} + \alpha\mu_m N_{m2} \\
f_{13}(Y(t)) &=& c_{hm} n \frac{I_{h1}^{\alpha}}{N_h^{\alpha}} S_{m1}^{\alpha} - (\sigma_m + \mu_m) E_{m1}^{\alpha} \\
f_{14}(Y(t)) &=& c_{hm} n \frac{I_{h2}^{\alpha}}{N_h^{\alpha}} S_{m2}^{\alpha} - (\sigma_m + \mu_m) E_{m2}^{\alpha} \\
f_{15}(Y(t)) &=& \sigma_m E_{m1}^{\alpha} - \mu_m I_{m1}^{\alpha} \\
f_{16}(Y(t)) &=& \sigma_m E_{m2}^{\alpha} - \mu_m I_{m2}^{\alpha}
\end{array}\right. \tag{38}$$

In condensed form, the system (38) can be written as:

$$\left\{\begin{array}{rcl}
\dot{Y}(t) &=& f(Y(t)) \\
Y(0) &=& (S_{h1}^{\alpha}(0), S_{h2}^{\alpha}(0), E_{h1}^{\alpha}(0), E_{h2}^{\alpha}(0), I_{h1}^{\alpha}(0), I_{h2}^{\alpha}(0), R_{h1}^{\alpha}(0), R_{h1}^{\alpha}(0), M_{h1}^{\alpha}(0), \\
&& M_{h2}^{\alpha}(0), S_{m1}^{\alpha}(0), S_{m2}^{\alpha}(0), E_{m1}^{\alpha}(0), E_{m2}^{\alpha}(0), I_{m1}^{\alpha}(0), I_{m2}^{\alpha}(0)).
\end{array}\right. \tag{39}$$

Theorem 2. *Given the initial conditions (33)-(34), the system (13)-(28) admits a unique, positive and bounded solution for all* $t \geq 0, \alpha \in [0,1]$ *in the positive invariant compact set* Ω *defined in (37).*

Proof. The functions on the right-hand side of system (13)-(28) are all $C^{\infty}-$functions, consequently $C^{1}-$functions. Therefore, the function f, as defined, is a differentiable function. Hence, by Wiggins & Golubitsky (1990), the function f is a local Lipschitz continued function in some opened balls containing $Y(0)$. It follows from Cauchy-Lipschitz Theorem (Wiggins & Golubitsky, 1990), the system (13)-(28) admits locally a unique solution.

In addition, suppose that $Y(t)$ is a solution of system (13)-(28) with $Y(0) \geq 0$, and let t_0 be the smallest positive real t such that $S^{\alpha}_{h1}(t_0) = 0$ or $S^{\alpha}_{h2}(t_0) = 0$ or $E^{\alpha}_{h1}(t_0) = 0$ or $E^{\alpha}_{h2}(t_0) = 0$ or $I^{\alpha}_{h1}(t_0) = 0$ or $I^{\alpha}_{h2}(t_0) = 0$ or $R^{\alpha}_{h1}(t_0) = 0$ or $R^{\alpha}_{h2}(t_0) = 0$ or $M^{\alpha}_{h1}(t_0) = 0$ or $M^{\alpha}_{h2}(t_0) = 0$ or $S^{\alpha}_{m1}(t_0) = 0$ or $S^{\alpha}_{m2}(t_0) = 0$ or $E^{\alpha}_{m1}(t_0) = 0$ or $E^{\alpha}_{m2}(t_0) = 0$ or $I^{\alpha}_{m1}(t_0) = 0$ or $I^{\alpha}_{m2}(t_0) = 0$. By continuity of functions $S^{\alpha}_{h1}, S^{\alpha}_{h2}, E^{\alpha}_{h1}, E^{\alpha}_{h2}, I^{\alpha}_{h1}, I^{\alpha}_{h2}, R^{\alpha}_{h1}, R^{\alpha}_{h2}, M^{\alpha}_{h1}, M^{\alpha}_{h2}, S^{\alpha}_{m1}, S^{\alpha}_{m2}, E^{\alpha}_{m1}, E^{\alpha}_{m2}, I^{\alpha}_{m1}$ and I^{α}_{m2} there exits $t^* > t_0$ such that if $S^{\alpha}_{h1}(t_0) = 0$, then from Equation (13), we have: $\dot{S}^{\alpha}_{h1}(t_0) = \Lambda^{\alpha}_{h1} + \eta R^{\alpha}_{h1}(t_0) \geq 0$. Therefore, for all $t \in [t_0, t^*]$, $S^{\alpha}_{h1}(t) \geq 0$. Consequently, S^{α}_{h1} is positive for all t. Similarly, we can establish the positivity of $S^{\alpha}_{h2}, E^{\alpha}_{h1}, E^{\alpha}_{h2}, I^{\alpha}_{h1}, I^{\alpha}_{h2}, R^{\alpha}_{h1}, R^{\alpha}_{h2}, M^{\alpha}_{h1}, M^{\alpha}_{h2}, S^{\alpha}_{m1}, S^{\alpha}_{m2}, E^{\alpha}_{m1}, E^{\alpha}_{m2}, I^{\alpha}_{m1}$ and I^{α}_{m2} for all $t \geq 0$.

The solutions $Y(t)$ of the model (13)-(28) are positives for all $t \geq 0$. Therefore, the feasible set Ω is positively invariant, consequently, for all $t \geq 0$, the solutions remain positives. Thus, the model (13)-(28) is well-posed mathematically and biologically meaningful. In addition, after some computations, one can show that:

$$\Omega = \Big\{(S^{\alpha}_{h1}, S^{\alpha}_{h2}, E^{\alpha}_{h1}, E^{\alpha}_{h2}, I^{\alpha}_{h1}, I^{\alpha}_{h2}, R^{\alpha}_{h1}, R^{\alpha}_{h2}, M^{\alpha}_{h1}, M^{\alpha}_{h1}, S^{\alpha}_{m1}, S^{\alpha}_{m2}, E^{\alpha}_{m1}, E^{\alpha}_{m2}, I^{\alpha}_{m1}, I^{\alpha}_{m2}) \in \mathbb{R}^{16}_{+} : N^{\alpha}_{h1}(t) \leq \frac{\Lambda^{\alpha}_{h1}}{\mu}, N^{\alpha}_{h2}(t) \leq \frac{\Lambda^{\alpha}_{h2}}{\mu}, N^{\alpha}_{m1}(t) \leq N^{\alpha}_{m1,0} \text{ and } N^{\alpha}_{m2}(t) \leq N^{\alpha}_{m2,0}\Big\}.$$

Finally, one can verify that the dissipation condition explained in (Caraballo and Han, 2016) is satisfied for system (13)-(28) and for all $t \geq 0$. That is:

$$f(Y).Y \quad \leq \quad a|Y|^2 + b;$$

where $a = 8\mu + 2\alpha c_{mh} n + 6\mu_m + 2c_{hm} n + \alpha\mu + 2(\sigma + \gamma + \delta + \eta + \varphi + \sigma_m)$ and
$b = [\Lambda^{\alpha}_{h1} + \Lambda^{\alpha}_{h2} + 2(\mu + \eta + 2c_{mh} n + \sigma + \varphi + \gamma)] (N^{\alpha}_{h})^2 + [2(\mu_m + c_{hm} n + \sigma_m)] (N^{\alpha}_{m})^2$.

Therefore, there exists a unique solution $Y(t)$ of system (13)-(28) globally defined in time. Additionally, since $S^{\alpha}_{h1}(t) \leq N^{\alpha}_{h1}$, $S^{\alpha}_{h2}(t) \leq N^{\alpha}_{h2}$, $E^{\alpha}_{h1}(t) \leq N^{\alpha}_{h1}$, $E^{\alpha}_{h2}(t) \leq N^{\alpha}_{h2}$, $I^{\alpha}_{h1}(t) \leq N^{\alpha}_{h1}$, $I^{\alpha}_{h2}(t) \leq N^{\alpha}_{h2}$, $R^{\alpha}_{h1}(t) \leq N^{\alpha}_{h1}$, $R^{\alpha}_{h2}(t) \leq N^{\alpha}_{h2}$, $M^{\alpha}_{h1}(t) \leq N^{\alpha}_{h1}$, $M^{\alpha}_{h2}(t) \leq N^{\alpha}_{h2}$, $S^{\alpha}_{m1}(t) \leq N^{\alpha}_{m1}$, $S^{\alpha}_{m2}(t) \leq N^{\alpha}_{m2}$, $E^{\alpha}_{m1}(t) \leq N^{\alpha}_{m1}$, $E^{\alpha}_{m2}(t) \leq N^{\alpha}_{m2}$, $I^{\alpha}_{m1}(t) \leq N^{\alpha}_{m1}$ and $I^{\alpha}_{m2}(t) \leq N^{\alpha}_{m2}$ for all $t \geq 0$; the solution $Y(t)$ is bounded. □

3.2 Equilibrium and fuzzy basic reproduction number

3.2.1 Equilibrium of the model

Theorem 3. *Consider the system (13)-(28) defined in the positive invariant compact set* Ω. *This system admits at least two equilibrium points.*

Proof. After some computations, one can show that, there are two equilibrium points. The first

one, called "disease-free equilibrium (DFE)", given by:

$$\begin{aligned} Y^0 &= (S_{h1}^{\alpha 0}, S_{h2}^{\alpha 0}, E_{h1}^{\alpha 0}, E_{h2}^{\alpha 0}, I_{h1}^{\alpha 0}, I_{h2}^{\alpha 0}, R_{h1}^{\alpha 0}, R_{h2}^{\alpha 0}, M_{h1}^{\alpha 0}, M_{h2}^{\alpha 0}, S_{m1}^{\alpha 0}, S_{m2}^{\alpha 0}, E_{m1}^{\alpha 0}, E_{m2}^{\alpha 0}, I_{m1}^{\alpha 0}, I_{m2}^{\alpha 0}) \\ &= \left(\frac{\Lambda_{h1}^{\alpha}}{\alpha\mu}, \frac{\Lambda_{h2}^{\alpha}}{\alpha\mu}, 0, 0, 0, 0, 0, 0, 0, 0, (1-\alpha)N_{m1} + \alpha N_{m2}, \right. \\ &\quad \left. (1-\alpha)N_{m3} + \alpha N_{m2}, 0, 0, 0, 0\right). \end{aligned} \tag{40}$$

This equilibrium occurs in the absence of malaria. In this situation, the whole human populations belong to the level set α of the fuzzy variable $\tilde{S}_h$; and the whole mosquito populations belong also to the level set α of the fuzzy variable $\tilde{S}_m$. The second one, called "endemic equilibrium (EE)", given by:

$$Y^* = (S_{h1}^{\alpha *}, S_{h2}^{\alpha *}, E_{h1}^{\alpha *}, E_{h2}^{\alpha *}, I_{h1}^{\alpha *}, I_{h2}^{\alpha *}, R_{h1}^{\alpha *}, R_{h2}^{\alpha *}, M_{h1}^{\alpha *}, M_{h2}^{\alpha *}, S_{m1}^{\alpha *}, S_{m2}^{\alpha *}, E_{m1}^{\alpha *}, E_{m2}^{\alpha *}, I_{m1}^{\alpha *}, I_{m2}^{\alpha *}), \tag{41}$$

where $S_{h1}^{\alpha *} = \frac{(\eta+\mu)(\varphi+\mu)\Lambda_{h1}^{\alpha}+\eta[\gamma\theta(\varphi+\mu)+\varphi\rho\gamma(1-\theta)]I_{h1}^{\alpha}}{\alpha(\eta+\mu)(\varphi+\mu)\left(c_{mh}n\frac{I_{m1}^{\alpha}}{N_h^{\alpha}}+\mu\right)}$, $S_{h2}^{\alpha} = \frac{(\eta+\mu)(\varphi+\mu)\Lambda_{h2}^{\alpha}+\eta[\gamma\theta(\varphi+\mu)+\varphi\rho\gamma(1-\theta)]I_{h2}^{\alpha}}{\alpha(\eta+\mu)(\varphi+\mu)\left(c_{mh}n\frac{I_{m2}^{\alpha}}{N_h^{\alpha}}+\mu\right)}$,

$E_{h1}^{\alpha} = \frac{c_{mh}n\frac{I_{m1}^{\alpha}}{N_h^{\alpha}}S_{h1}^{\alpha}}{\sigma+\mu}$, $E_{h2}^{\alpha} = \frac{c_{mh}n\frac{I_{m2}^{\alpha}}{N_h^{\alpha}}S_{h2}^{\alpha}}{\sigma+\mu}$, $I_{h1}^{\alpha} = \frac{\sigma(\varphi+\mu)c_{mh}nI_{m1}^{\alpha}S_{h1}^{\alpha}}{(\sigma+\mu)[(\varphi+\mu)(\gamma+\mu+\delta)-\varphi\gamma(1-\rho)(1-\theta)]N_h^{\alpha}}$,

$R_{h1}^{\alpha} = \frac{[\gamma\theta(\varphi+\mu)+\varphi\rho\gamma(1-\theta)]I_{h1}^{\alpha}}{(\eta+\mu)(\varphi+\mu)}$, $I_{h2}^{\alpha} = \frac{\sigma(\varphi+\mu)c_{mh}nI_{m2}^{\alpha}S_{h2}^{\alpha}}{(\sigma+\mu)[(\varphi+\mu)(\gamma+\mu+\delta)-\varphi\gamma(1-\rho)(1-\theta)]N_h^{\alpha}}$,

$R_{h2}^{\alpha} = \frac{[\gamma\theta(\varphi+\mu)+\varphi\rho\gamma(1-\theta)]I_{h2}^{\alpha}}{(\eta+\mu)(\varphi+\mu)}$, $M_{h1}^{\alpha} = \frac{\gamma(1-\theta)}{\varphi+\mu}I_{h1}^{\alpha}$, $M_{h2}^{\alpha} = \frac{\gamma(1-\theta)}{\varphi+\mu}I_{h2}^{\alpha}$, $S_{m1}^{\alpha} = \frac{(1-\alpha)\mu_m N_{m1}+\alpha\mu_m N_{m2}}{c_{hm}n\frac{I_{h1}^{\alpha}}{N_h^{\alpha}}+\mu_m}$,

$S_{m2}^{\alpha} = \frac{(1-\alpha)\mu_m N_{m3}+\alpha\mu_m N_{m2}}{c_{hm}n\frac{I_{h2}^{\alpha}}{N_h^{\alpha}}+\mu_m}$, $E_{m1}^{\alpha} = \frac{c_{hm}n((1-\alpha)\mu_m N_{m1}+\alpha\mu_m N_{m2})I_{h1}^{\alpha}}{(\sigma_m+\mu_m)[c_{hm}nI_{h1}^{\alpha}+\mu_m N_h^{\alpha}]}$, $E_{m2}^{\alpha} = \frac{c_{hm}n((1-\alpha)\mu_m N_{m3}+\alpha\mu_m N_{m2})I_{h2}^{\alpha}}{(\sigma_m+\mu_m)[c_{hm}nI_{h2}^{\alpha}+\mu_m N_h^{\alpha}]}$,

$I_{m1}^{\alpha} = \frac{\sigma_m E_{m1}^{\alpha}}{\mu_m}$ and $I_{m2}^{\alpha} = \frac{\sigma_m E_{m2}^{\alpha}}{\mu_m}$. □

3.2.2 Fuzzy basic reproduction number, $\mathcal{R}_0^{\alpha}$

In this part, we introduce a new definition of fuzzy basic reproduction number depending on the level α, noted by $\mathcal{R}_0^{\alpha}$. This threshold measures the average number of new malaria contamination that one infectious individual can produce when introduced into an healthy population during his infectious period. We use the well-known algorithm, the next generation matrix's algorithm, as explained in (van den Driessche & Watmough, 2002) to compute the fuzzy basic reproduction number. The compartments $E_{h1}^{\alpha}, E_{h2}^{\alpha}, I_{h1}^{\alpha}, I_{h2}^{\alpha}, M_{h1}^{\alpha}, M_{h2}^{\alpha}$, $E_{m1}^{\alpha}, E_{m2}^{\alpha}, I_{m1}^{\alpha}$ and I_{m2}^{α} of the model (13)-(28) are considered to be the disease compartments, and $S_{h1}^{\alpha}, S_{h2}^{\alpha}, R_{h1}^{\alpha}, R_{h2}^{\alpha}, S_{m1}^{\alpha}$, and S_{m2}^{α} are the non-disease compartments. Setting $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6, \mathcal{F}_7, \mathcal{F}_8, \mathcal{F}_9, \mathcal{F}_{10})^T$ and $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5, \mathcal{V}_6, \mathcal{V}_7, \mathcal{V}_8, \mathcal{V}_9, \mathcal{V}_{10})^T$, where $\mathcal{F}_i$ represents the rate of new infections in the i^{th} disease compartment, $\mathcal{V}_i^+$ is the transfer rate of individuals into compartment i by all other means while $\mathcal{V}_i^-$ represents the transfer rate of individual out of compartment i, we have these two matrices:

$$\mathcal{F} = \begin{pmatrix} c_{mh}n\frac{I_{m1}^{\alpha}}{N_h^{\alpha}}S_{h1}^{\alpha} \\ c_{mh}n\frac{I_{m2}^{\alpha}}{N_h^{\alpha}}S_{h2}^{\alpha} \\ 0 \\ 0 \\ 0 \\ 0 \\ c_{hm}n\frac{I_{h1}^{\alpha}}{N_h^{\alpha}}S_{m1}^{\alpha} \\ c_{hm}n\frac{I_{h2}^{\alpha}}{N_h^{\alpha}}S_{m2}^{\alpha} \\ 0 \\ 0 \end{pmatrix} \quad and \quad \mathcal{V} = \begin{pmatrix} (\sigma+\mu)E_{h1}^{\alpha} \\ (\sigma+\mu)E_{h2}^{\alpha} \\ (\gamma+\mu+\delta)I_{h1}^{\alpha} - \sigma E_{h1}^{\alpha} - \varphi(1-\rho)M_{h1}^{\alpha} \\ (\gamma+\mu+\delta)I_{h2}^{\alpha} - \sigma E_{h2}^{\alpha} - \varphi(1-\rho)M_{h2}^{\alpha} \\ (\varphi+\mu)M_{h1}^{\alpha} - \gamma(1-\theta)I_{h1}^{\alpha} \\ (\varphi+\mu)M_{h2}^{\alpha} - \gamma(1-\theta)I_{h2}^{\alpha} \\ (\sigma_m+\mu_m)E_{m1}^{\alpha} \\ (\sigma_m+\mu_m)E_{m2}^{\alpha} \\ \mu_m I_{m1}^{\alpha} - \sigma_m E_{m1}^{\alpha} \\ \mu_m I_{m2}^{\alpha} - \sigma_m E_{m2}^{\alpha} \end{pmatrix}$$

The Jacobians of matrices $\mathcal{F}$ and $\mathcal{V}$ evaluate at the DFE are: $F = [F_{ij}]_{10\times 10}$, where $F_{73} = c_{hm} n \frac{(1-\alpha)N_{m1}+\alpha N_{m2}}{N_h^\alpha}$, $F_{84} = c_{hm} n \frac{(1-\alpha)N_{m3}+\alpha N_{m2}}{N_h^\alpha}$, $F_{19} = \frac{c_{mh} n \Lambda_{h1}^\alpha}{\alpha\mu N_h^\alpha}$, $F_{2,10} = \frac{c_{mh} n \Lambda_{h2}^\alpha}{\alpha\mu N_h^\alpha}$ and $F_{ij} = 0$, for all other i, j; and

$$V = \begin{pmatrix} k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\sigma & 0 & k_2 & 0 & -k_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sigma & 0 & k_2 & 0 & -k_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & -k_3 & 0 & k_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_3 & 0 & k_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sigma_m & 0 & \mu_m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma_m & 0 & \mu_m \end{pmatrix},$$

where:

$$k_1 = \sigma + \mu, k_2 = \gamma + \mu + \delta, k_3 = \gamma(1-\theta), k_4 = \varphi(1-\rho), k_5 = \varphi + \mu \text{ and } k_6 = \sigma_m + \mu_m. \quad (42)$$

The fuzzy basic reproduction number depending on level α, is given by:

$$\mathcal{R}_0^\alpha = \rho(FV^{-1}) = n\sqrt{\frac{\Lambda_{h2}^\alpha c_{hm} c_{mh} k_5 \sigma\sigma_m}{\alpha k_1 k_6 \mu\mu_m (k_2k_5 - k_3k_4)} \frac{(1-\alpha)N_{m3} + \alpha N_{m2}}{(N_h^\alpha)^2}}, \quad (43)$$

with $k_2k_5 - k_3k_4 = \varphi(\mu+\delta) + \mu(\gamma + \mu^2 + \delta) + \gamma\rho\varphi(1-\theta) > 0$. Where $\rho(.)$ represents the spectral radius of the next generation matrix FV^{-1}.

When $\mathcal{R}_0^\alpha < 1$, malaria disappears in the population and when $\mathcal{R}_0^\alpha > 1$, malaria continues to spread in the population and becomes endemic. The fraction $\frac{(1-\alpha)N_{m3}+\alpha N_{m2}}{(N_h^\alpha)^2}$ represents the weighted proportion of mosquitoes in the entire level set of human population. The ratios $1/k_1$ and $1/k_6$ represent the latent periods of human and mosquito populations respectively. With a specific level $\alpha \in]0, 1]$, we have the corresponding related fuzzy basic reproduction number. Notice that the level α is inversely proportional to the fuzzy basic reproduction number $\mathcal{R}_0^\alpha$. That is, as α increases, $\mathcal{R}_0^\alpha$ decreases.

3.3 Stability of the Disease-Free Equilibrium

Theorem 4. *The disease free equilibrium* $Y^0 = \left(\frac{\Lambda_{h1}^\alpha}{\alpha\mu}, \frac{\Lambda_{h2}^\alpha}{\alpha\mu}, 0, 0, 0, 0, 0, 0, 0, 0, (1-\alpha)N_{m1} + \alpha N_{m2}, (1-\alpha)N_{m3} + \alpha N_{m2}, 0, 0, 0, 0\right)$ *of system (13)-(28) is globally asymptotically stable in the invariant positive compact set* Ω *if* $\mathcal{R}_0^\alpha \leq 1$ *for all* $\alpha \in]0, 1]$. *If* $\mathcal{R}_0^\alpha > 1$, *the disease free equilibrium is unstable, the system is uniformly persistent and there exists at least one equilibrium point in the interior of* Ω.

Proof. For the construction of the Lyapunov function, we use the matrix-theoretical method as explained in (Shuai & van den Driessche, 2013). Assume that
$x = (E_{h1}^\alpha, E_{h2}^\alpha, I_{h1}^\alpha, I_{h2}^\alpha, M_{h1}^\alpha, M_{h2}^\alpha, E_{m1}^\alpha, E_{m2}^\alpha, I_{m1}^\alpha, I_{m2}^\alpha)^T$ and $y = (S_{h1}^\alpha, S_{h2}^\alpha, R_{h1}^\alpha, R_{h2}^\alpha, S_{m1}^\alpha, S_{m2}^\alpha)^T$.
The computations of $f(x, y) = (F - V)x - \mathcal{F}(x, y) + \mathcal{V}(x, y)$ give:

$$f(x,y) = (F-V)x - \mathcal{F}(x,y) + \mathcal{V}(x,y) = \begin{pmatrix} c_{mh}nI^{\alpha}_{m1}\left(\frac{\Lambda^{\alpha}_{h1}}{\alpha\mu N^{\alpha}_{h}} - \frac{S^{\alpha}_{h1}}{N^{\alpha}_{h}}\right) \\ c_{mh}nI^{\alpha}_{m2}\left(\frac{\Lambda^{\alpha}_{h2}}{\alpha\mu N^{\alpha}_{h}} - \frac{S^{\alpha}_{h2}}{N^{\alpha}_{h}}\right) \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{c_{hm}nI^{\alpha}_{h1}}{N^{\alpha}_{h}}[(1-\alpha)N_{m1} + \alpha N_{m2} - S^{\alpha}_{m1}] \\ \frac{c_{hm}nI^{\alpha}_{h2}}{N^{\alpha}_{h}}[(1-\alpha)N_{m3} + \alpha N_{m2} - S^{\alpha}_{m2}] \\ 0 \\ 0 \end{pmatrix}.$$

Since $S^{\alpha}_{h1} \leq N^{\alpha}_{h}, S^{\alpha}_{h2} \leq N^{\alpha}_{h}, S^{\alpha}_{m1} \leq (1-\alpha)N_{m1} + \alpha N_{m2}$ and $S^{\alpha}_{m2} \leq (1-\alpha)N_{m3} + \alpha N_{m2}$, then $f(x,y) \geq 0$. Notice again that $F \geq 0, V^{-1} \geq 0$ and $f(x,y^T) = 0$ in the feasible set Ω. Therefore, since the matrix $V^{-1}F$ is reducible, we use Theorem 2.1 in (Shuai & van den Driessche, 2013) to construct Lyapunov function of system (13)-(28). For this construction, let $\omega^T = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}) \geq 0$ be the left eigenvector of the non-negative matrix $V^{-1}F$ corresponding to the eigenvalue $\mathcal{R}^{\alpha}_{0}$. Then:

$$(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10})V^{-1}F = \mathcal{R}^{\alpha}_{0}(v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}) \quad (44)$$

From Equation (44), we obtain: $v_1 = v_2 = v_5 = v_6 = v_7 = v_8 = 0$,
$v_3 = \dfrac{c_{hm}n[(1-\alpha)N_{m1} + \alpha N_{m2}]\sigma_m}{\mathcal{R}^{\alpha}_{0}k_6\mu_m N^{\alpha}_{h}}v_9, v_9 \in \mathbb{R}^+$ and $v_4 = \dfrac{c_{hm}n[(1-\alpha)N_{m3} + \alpha N_{m2}]\sigma_m}{\mathcal{R}^{\alpha}_{0}k_6\mu_m N^{\alpha}_{h}}v_{10}$,
where $v_{10} \in \mathbb{R}^+$. Therefore,
$\omega^T = \left(0, 0, \dfrac{c_{hm}n[(1-\alpha)N_{m1} + \alpha N_{m2}]\sigma_m}{\mathcal{R}^{\alpha}_{0}k_6\mu_m N^{\alpha}_{h}}v_9, \dfrac{c_{hm}n[(1-\alpha)N_{m3} + \alpha N_{m2}]\sigma_m}{\mathcal{R}^{\alpha}_{0}k_6\mu_m N^{\alpha}_{h}}v_{10}, 0, 0, 0, 0, v_9, v_{10}\right)$.
By Theorem 2.1 of (Shuai & van den Driessche, 2013), the function L defined by:

$$\begin{aligned} L &= \omega^T V^{-1}x \\ &= \left[\frac{c_{hm}n\sigma\sigma_m k_5}{\mathcal{R}^{\alpha}_{0}k_1k_2k_6\mu_m N^{\alpha}_{h}(1-k_3k_4)}\left[((1-\alpha)N_{m1} + \alpha N_{m2})v_9E^{\alpha}_{h1} + ((1-\alpha)N_{m3} + \alpha N_{m2})v_{10}E^{\alpha}_{h2}\right]\right. \\ &+ \frac{\sigma_m}{\mu_m k_6}(v_9E^{\alpha}_{m1} + v_{10}E^{\alpha}_{m2}) + \frac{c_{hm}n[(1-\alpha)N_{m1} + \alpha N_{m2}]\sigma_m}{\mathcal{R}^{\alpha}_{0}k_6\mu_m N^{\alpha}_{h}(k_2k_5 - k_3k_4)}(k_5I^{\alpha}_{h1} + k_4M^{\alpha}_{h1})v_9 \\ &+ \left.\frac{c_{hm}n[(1-\alpha)N_{m3} + \alpha N_{m2}]\sigma_m}{\mathcal{R}^{\alpha}_{0}k_6\mu_m N^{\alpha}_{h}(k_2k_5 - k_3k_4)}(k_5I^{\alpha}_{h2} + k_4M^{\alpha}_{h2})v_{10} + \frac{1}{\mu_m}(v_9I^{\alpha}_{m1} + v_{10}I^{\alpha}_{m2})\right], v_9 > 0 \text{ and } v_{10} > 0, \\ &\forall\, Y \neq Y^0, Y \in \Omega; \end{aligned}$$

is a candidate Lyapunov function for system (13)-(28) with k_1, k_2, k_3, k_4, k_5 and k_6 defined in

(42). The differentiation of L with respect to t gives:

$$\begin{aligned}
\dot{L} &= (\mathcal{R}_0^\alpha - 1)\omega^T x - \omega^T V^{-1} f(x,y) \\
&= (\mathcal{R}_0^\alpha - 1)\left[\frac{c_{hm} n \sigma_m}{\mathcal{R}_0^\alpha k_6 \mu_m N_h^\alpha}[((1-\alpha)N_{m1} + \alpha N_{m2})v_9 I_{h1}^\alpha \right. \\
&+ \left. ((1-\alpha)N_{m3} + \alpha N_{m2})v_{10} I_{h2}^\alpha] + v_9 I_{m1}^\alpha + v_{10} I_{m2}^\alpha\right] \\
&- \frac{c_{hm} c_{mh} n^2 \sigma \sigma_m k_5 [(1-\alpha)N_{m1} + \alpha N_{m2}]}{\mathcal{R}_0^\alpha k_6 \mu_m k_1 k_2 (1 - k_3 k_4) N_h^\alpha} \left(\frac{\Lambda_{h1}^\alpha}{\alpha \mu N_h^\alpha} - \frac{S_{h1}^\alpha}{N_h^\alpha}\right) v_9 I_{m1}^\alpha \\
&- \frac{c_{hm} c_{mh} n^2 \sigma \sigma_m k_5 [(1-\alpha)N_{m3} + \alpha N_{m2}]}{\mathcal{R}_0^\alpha k_6 \mu_m k_1 k_2 (1 - k_3 k_4) N_h^\alpha} \left(\frac{\Lambda_{h2}^\alpha}{\alpha \mu N_h^\alpha} - \frac{S_{h2}^\alpha}{N_h^\alpha}\right) v_{10} I_{m2}^\alpha \\
&- \frac{c_{hm} n \sigma_m}{k_6 \mu_m N_h^\alpha}[(1-\alpha)N_{m1} + \alpha N_{m2} - S_{m1}^\alpha] v_9 I_{h1}^\alpha \\
&- \frac{c_{hm} n \sigma_m}{k_6 \mu_m N_h^\alpha}[(1-\alpha)N_{m3} + \alpha N_{m2} - S_{m2}^\alpha] v_{10} I_{h2}^\alpha. \qquad (45)
\end{aligned}$$

Since $S_{h1}^\alpha \leq N_h^\alpha, S_{h2}^\alpha \leq N_h^\alpha, S_{m1}^\alpha \leq (1-\alpha)N_{m1} + \alpha N_{m2}$ and $S_{m2}^\alpha \leq (1-\alpha)N_{m3} + \alpha N_{m2}$, if $\mathcal{R}_0^\alpha \leq 1$, then $\dot{L} < 0$, which implies that L is a Lyapunov function for system (13)-(28). Furthermore, $\dot{L} = 0$ implies that $I_{h1}^\alpha = I_{h2}^\alpha = I_{m1}^\alpha = I_{m2}^\alpha = 0, S_{h1}^\alpha = \frac{\Lambda_{h1}^\alpha}{\alpha\mu}, S_{h2}^\alpha = \frac{\Lambda_{h2}^\alpha}{\alpha\mu}, S_{m1}^\alpha = (1-\alpha)N_{m1} + \alpha N_{m2}$ and $S_{m2}^\alpha = (1-\alpha)N_{m3} + \alpha N_{m2}$. Thus, the largest invariant set of the model (13)-(28) when $\dot{L} = 0$ in the interior of the feasible set Ω is the singleton $\{Y^0\}$. Therefore, by the LaSalle's invariance principle (LaSalle, 1976), the disease-free equilibrium Y^0 is globally asymptotically stable provided that $\mathcal{R}_0^\alpha \leq 1$. for all $\alpha \in]0,1]$. In addition, if $\mathcal{R}_0^\alpha > 1$, then $\dot{L} > 0$ for $S_{h1}^\alpha = \frac{\Lambda_{h1}^\alpha}{\alpha\mu}, S_{h2}^\alpha = \frac{\Lambda_{h2}^\alpha}{\alpha\mu}, S_{m1}^\alpha = (1-\alpha)N_{m1} + \alpha N_{m2}$ and $S_{m2}^\alpha = (1-\alpha)N_{m3} + \alpha N_{m2}$. Therefore, by continuity, $\dot{L}$ remains positive in a small neighborhood of the disease-free equilibrium Y^0, implies that Y^0 is unstable when $\mathcal{R}_0 > 1$, and using Theorem 2.1 of (Shuai & van den Driessche, 2013), the system (13)-(28) is uniformly persistent, implies that there exists at least one endemic equilibrium of system (13)-(28), noted by Y^*, defined by (41) in the interior of the feasible set Ω. □

3.4 Stability of the Endemic Equilibrium

Theorem 5. *The endemic equilibrium $Y^* \in int(\Omega)$ defined in (41) of the system (13)-(28) is globally asymptotically stable in the interior of the positively-invariant compact set Ω if $\mathcal{R}_0^\alpha > 1$, with $\alpha \in]0,1]$.*

Proof. We use the graph-theoretical method as explained in (Shuai & van den Driessche, 2013). For the construction of a lyapunov function, let:

$$\begin{aligned}
L_1 &= S_{h1}^\alpha - S_{h1}^{\alpha*} - S_{h1}^{\alpha*}\ln(S_{h1}/S_{h1}^{\alpha*})\ ;\ L_2 = S_{h2}^\alpha - S_{h2}^{\alpha*} - S_{h2}^{\alpha*}\ln(S_{h2}/S_{h2}^{\alpha*}) \\
L_3 &= E_{h1}^\alpha - E_{h1}^{\alpha*} - E_{h1}^{\alpha*}\ln(E_{h1}/E_{h1}^{\alpha*})\ ;\ L_4 = E_{h2}^\alpha - E_{h2}^{\alpha*} - E_{h2}^{\alpha*}\ln(E_{h2}/E_{h2}^{\alpha*}) \\
L_5 &= I_{h1}^\alpha - I_{h1}^{\alpha*} - I_{h1}^{\alpha*}\ln(I_{h1}/I_{h1}^{\alpha*})\ ;\ L_6 = I_{h2}^\alpha - I_{h2}^{\alpha*} - I_{h2}^{\alpha*}\ln(I_{h2}/I_{h2}^{\alpha*}) \\
L_7 &= R_{h1}^\alpha - R_{h1}^{\alpha*} - R_{h1}^{\alpha*}\ln(R_{h1}/R_{h1}^{\alpha*})\ ;\ L_8 = R_{h2}^\alpha - R_{h2}^{\alpha*} - R_{h2}^{\alpha*}\ln(R_{h2}/R_{h2}^{\alpha*}) \\
L_9 &= M_{h1}^\alpha - M_{h1}^{\alpha*} - M_{h1}^{\alpha*}\ln(M_{h1}/M_{h1}^{\alpha*})\ ;\ L_{10} = M_{h2}^\alpha - M_{h2}^{\alpha*} - M_{h2}^{\alpha*}\ln(M_{h2}/M_{h2}^{\alpha*}) \\
L_{11} &= S_{m1}^\alpha - S_{m1}^{\alpha*} - S_{m1}^{\alpha*}\ln(S_{m1}/S_{m1}^{\alpha*})\ ;\ L_{12} = S_{m2}^\alpha - S_{m2}^{\alpha*} - S_{m2}^{\alpha*}\ln(S_{m2}/S_{m2}^{\alpha*}) \\
L_{13} &= E_{m1}^\alpha - E_{m1}^{\alpha*} - E_{m1}^{\alpha*}\ln(E_{m1}/E_{m1}^{\alpha*})\ ;\ L_{14} = E_{m2}^\alpha - E_{m2}^{\alpha*} - E_{m2}^{\alpha*}\ln(E_{m2}/E_{m2}^{\alpha*}) \\
L_{15} &= I_{m1}^\alpha - I_{m1}^{\alpha*} - I_{m1}^{\alpha*}\ln(I_{m1}/I_{m1}^{\alpha*}) \text{ and } L_{16} = I_{m2}^\alpha - I_{m2}^{\alpha*} - I_{m2}^{\alpha*}\ln(I_{m2}/I_{m2}^{\alpha*})
\end{aligned}$$

The weights of digraph generated by the derivatives of functions $L_i, \forall i \in \{1,2,3,\ldots,16\}$ with respect to t are: $a_{1,15} = \alpha c_{mh} n \frac{I_{m1}^{\alpha*}}{N_h^\alpha} S_{h1}^{\alpha*}, a_{17} = \eta R_{h1}^{\alpha*}, a_{2,16} = \alpha c_{mh} n \frac{I_{m2}^{\alpha*}}{N_h^\alpha} S_{h2}^{\alpha*}, a_{28} = \eta R_{h2}^{\alpha*}, a_{31} =$

$c_{mh}n\frac{I^{\alpha*}_{m1}}{N^{\alpha}_{h}}S^{\alpha*}_{h1}, a_{42} = c_{mh}n\frac{I^{\alpha*}_{m2}}{N^{\alpha}_{h}}S^{\alpha*}_{h2}, a_{53} = \sigma E^{\alpha*}_{h1}, a_{59} = \varphi(1-\rho)M^{\alpha*}_{h1}, a_{64} = \sigma E^{\alpha*}_{h2}, a_{6,10} = \varphi(1-\rho)M^{\alpha*}_{h2}, a_{75} = \gamma\theta I^{\alpha*}_{h1}, a_{79} = \varphi\rho M^{\alpha*}_{h1}, a_{86} = \gamma\theta I^{\alpha*}_{h2}, a_{8,10} = \varphi\rho M^{\alpha*}_{h2}, a_{95} = \gamma(1-\theta)I^{\alpha*}_{h1}, a_{10,6} = \gamma(1-\theta)I^{\alpha*}_{h2}, a_{11,5} = a_{13,5} = c_{hm}n\frac{I^{\alpha*}_{h1}}{N^{\alpha}_{h}}S^{\alpha*}_{m1},\ a_{12,6} = a_{14,6} = c_{hm}n\frac{I^{\alpha*}_{h2}}{N^{\alpha}_{h}}S^{\alpha*}_{m2}, a_{15,13} = \sigma_m E^{\alpha*}_{m1}$ *and* $a_{16,14} = \sigma_m E^{\alpha*}_{m2}$; and all other $a_{ij} = 0$ such that the weighted matrix is $A = [a_{ij}]_{16\times16}$, where $a_{ij} > 0$, (the weight of arc (j, i)). This implies that the first condition of Theorem 3.5 in (Shuai & van den Driessche, 2013) is satisfied. One can check that in each directed cycle of the graph (G, A), we have: $G_{71} + G_{13} + G_{35} + G_{57} = G_{10,6} + G_{6,10} = G_{82} + G_{24} + G_{46} + G_{68} = G_{13} + G_{35} + G_{5,13} + G_{13,15} + G_{15,1} = G_{14,16} + G_{16,2} + G_{24} + G_{46} + G_{6,14} = G_{59} + G_{95} = 0$. This means that the second condition of Theorem 3.5 in (Shuai & van den Driessche, 2013) is also satisfied, therefore, there exists $c_{i,i\in\{1,2,\ldots,15,16\}}$ such that:

$$L = \sum_{i=1}^{16} c_i L_i, \tag{46}$$

is a Lyapunov function for system (13)-(28). The relations between c_i are therefore derived from theorems 3.3 and 3.4 of (Shuai & van den Driessche, 2013) as follow: $c_7 = c_1a_{17}/(a_{75} + a_{79}), c_3 = c_1(a_{17} + a_{1,15})/a_{31}, c_5 = c_3a_{31}/a_{53}, c_2 = c_4a_{42}/(a_{28} + a_{2,16}), c_6 = c_4a_{42}/a_{64}, c_{16} = c_2a_{2,16}/a_{16,14}, c_1 = c_{15}a_{15,13}/a_{1,15}, c_{11} = c_{12} = 0, c_8 = c_2a_{28}/(a_{86}+a_{8,10}), c_{14} = c_{16}a_{16,14}/a_{14,16}, c_{15} = c_{13}a_{13,5}/a_{15,13}, c_9 = (c_5a_{59} + c_7a_{79})/a_{95}, c_{10} = (c_6a_{6,10} + c_8a_{8,10})/a_{10,6}$. From the values of c_i, Relation (46) becomes:

$$L = \sum_{i=1}^{10} c_i L_i + \sum_{i=13}^{16} c_i L_i. \tag{47}$$

The fact that $\dot{L} = \sum_{i=1}^{16} c_i\dot{L}_i \leq 0$ implies that $Y = Y^*$. Hence, the largest invariant set for system (13)-(28), where $\dot{L} = 0$ is the singleton set $\{Y^*\}$. This proves the uniqueness and global asymptotic stability of Y^*in the interior of Ω provided that $\mathcal{R}_0 > 1$. □

4 Numerical analysis

In this section, we present the numerical results of the model (13)-(28). It is very important to notice that when the level $\alpha = 1$, we obtain the same results as in (Mangongo et al., 2022). This proves the flexibility of fuzzy models, comparing with the deterministic one. This flexibility gives us possibility to do several analysis for different level α. After presenting the values of parameters and initial conditions of state variables, we start first by the local sensitivity analysis of the model and we end the section by some numerical simulations for different level α.

4.1 Parameter values and initial conditions for state variables

All parameter values used in this paper are summarized in Table 1. The lifespan of an *Anopheles* mosquito is estimated in the range of 13 to 23 days (Ndoen et al., 2012). Taking the inverse of the upper bound of it, we obtain the natural death rate of mosquito $\mu_m = 435 \times 10^{-4}$. The lifespan of *Plasmodium* species depends on each type of them, but in most of cases, it ranges between 12 to 14 days (Adugna et al., 2022). Taking the inverse of upper bound of this range, we obtain the incubation rate of mosquito $\sigma_m = 71 \times 10^{-3}$. About the initial conditions of the fuzzy state variables, we suppose that: $\tilde{S}_{m0} = (50, 100, 150)$, $\tilde{E}_{m0} = (25, 50, 75)$ and $\tilde{I}_{m0} = (75, 100, 125)$. Adding them, we get the fuzzy initial population of mosquitoes, which is given by $\tilde{N}_{m0} = (N_{m01}, N_{m02}, N_{m03}) = (150, 250, 350)$. For the crisp states variables, we suppose that $S_{h0} = 200, E_{h0} = 100, I_{h0} = 100, R_{h0} = 50$ and $M_{h0} = 50$. The fuzzification of the initial conditions of the crisp state variables gives: $\tilde{S}_{h0} = (150, 200, 300), \tilde{E}_{h0} = (75, 100, 150), \tilde{I}_{h0} = (75, 100, 150), \tilde{R}_{h0} = (30, 50, 90)$ and $\tilde{M}_{h0} = (30, 50, 90)$, which are triangular fuzzy variables. The level sets α of our initial conditions are given by: $S^{\alpha}_{h0} = [S^{\alpha}_{h01}, S^{\alpha}_{h02}] = [150 + 50\alpha, 300 - 100\alpha], E^{\alpha}_{h0} = [E^{\alpha}_{h01}, E^{\alpha}_{h02}] = [75 + 25\alpha, 150 - 50\alpha], I^{\alpha}_{h0} = [I^{\alpha}_{h01}, I^{\alpha}_{h02}] = [75 +$

$25\alpha, 150-50\alpha]$, $R_{h0}^{\alpha}=[R_{h01}^{\alpha}, R_{h02}^{\alpha}]=[30+20\alpha, 90-40\alpha]$, $M_{h0}^{\alpha}=[M_{h01}^{\alpha}, M_{h02}^{\alpha}]=[30+20\alpha, 90-40\alpha]$, $S_{m0}^{\alpha}=[S_{m01}^{\alpha}, S_{m02}^{\alpha}]=[50+50\alpha, 150-50\alpha]$, $E_{m0}^{\alpha}=[E_{m01}^{\alpha}, E_{m02}^{\alpha}]=[25+25\alpha, 75-25\alpha]$ and $I_{m0}^{\alpha}=[I_{m01}^{\alpha}, I_{m02}^{\alpha}]=[75+25\alpha, 125-25\alpha]$. For a specific level $\alpha \in [0,1]$, we can find the specifics corresponding level sets α for each state variable. The fuzzification of constant recruitment Λ_h into a triangular fuzzy variable can be $\tilde{\Lambda}_h = (0.5; 1; 1.5)$. The level set α of $\tilde{\Lambda}_h$ is therefore $\tilde{\Lambda}_h^{\alpha} = [0.5+0.5\alpha; 1.5-0.5\alpha]$.

4.2 Local sensitivity analysis

The local sensitivity analysis gives an idea of some control parameters that are very important to reduce very significantly $\mathcal{R}_0^{\alpha}$ in order to control malaria. The threshold $\mathcal{R}_0^{\alpha}$ is a very important tool in the mathematical analysis of the spread of a disease, therefore, it's very important to analyze its sensitivity according to each control parameter compose it. We calculate first $\frac{\partial \mathcal{R}_0^{\alpha}}{\partial \varkappa}\frac{\varkappa}{\mathcal{R}_0^{\alpha}}$ for each parameter $\varkappa$ using the baseline values given in Table 1. These calculations lead us to find the sensitivity indices, which measure the ratio of relative change in $\mathcal{R}_0^{\alpha}$ to the relative change in parameter $\varkappa$. See Table 2. We can see from that Table the derivatives of $\mathcal{R}_0^{\alpha}$ do not depend on the level α.

Table 2: Sensitivity indices of $\mathcal{R}_0^{\alpha}$ relative to some control parameters of the model

Parameters	**Formula:** $\frac{\partial \mathcal{R}_0^{\alpha}}{\partial \varkappa} * \frac{\varkappa}{\mathcal{R}_0^{\alpha}}$	**Values**	**Ranges**	**Indices**
n	1	4	1-10	1
γ	$-\frac{\gamma(k_5-(1-\theta)k_4)}{2(k_2k_5-k_3k_4)}$	0.071	0.002-0.2	-0.5
θ	$-\frac{\gamma\theta k_4}{2(k_2k_5-k_3k_4)}$	0.2	0.01-1	-0.5
φ	$-\frac{\varphi k_3((1-\rho)k_5-k_4)}{2k_5(k_2k_5-k_3k_4)}$	0.1	0.005-0.5	0.0008098

The average number of mosquito bites takes the top of the list with sensitivity index 1. The proportion θ of humans who recover and the recovery rate γ come in the second position with sensitivity index -0.5, followed by the relapse rate whose the sensitivity index is 0.0008098. This means that, the average number of mosquito bites influences very significantly the spread of malaria in the population. This implies the well-known policy measures, like the use of bed nets and spray of insecticides for killing mosquitoes.

4.3 Numerical simulations

In this section, we realize some numerical simulations of the model (13)-(28) for different levels $\alpha, \alpha = 0.5$ and $\alpha = 0.9$. From Figures 2 and 3, we see the endemic trends of both human and mosquito populations respectively for $\alpha = 0.5$.

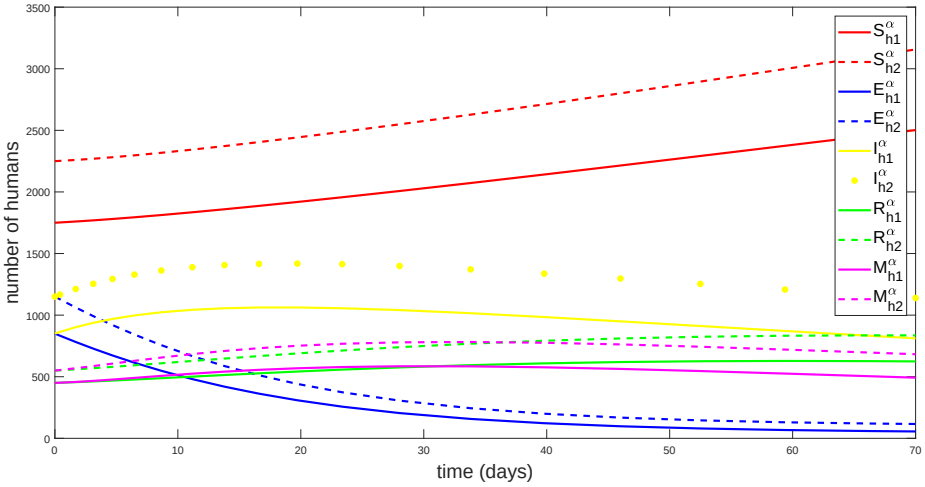

Figure 2: Endemic trends of human populations at level $\alpha = 0,5$ with parameter values given in Table 1 and initial conditions given in section 4.1. The corresponding fuzzy basic reproduction number and the estimated endemic state for human populations are $\mathcal{R}_0^{\alpha} = 3.057$ and $(S^{\alpha}_{h1}, S^{\alpha}_{h2}, E^{\alpha}_{h1}, E^{\alpha}_{h2}, I^{\alpha}_{h1}, I^{\alpha}_{h2}, R^{\alpha}_{h1}, R^{\alpha}_{h2}, M^{\alpha}_{h1}, M^{\alpha}_{h2}) = (2502, 3157, 56, 116, 836, 1139, 624, 836, 492, 683)$ respectively.

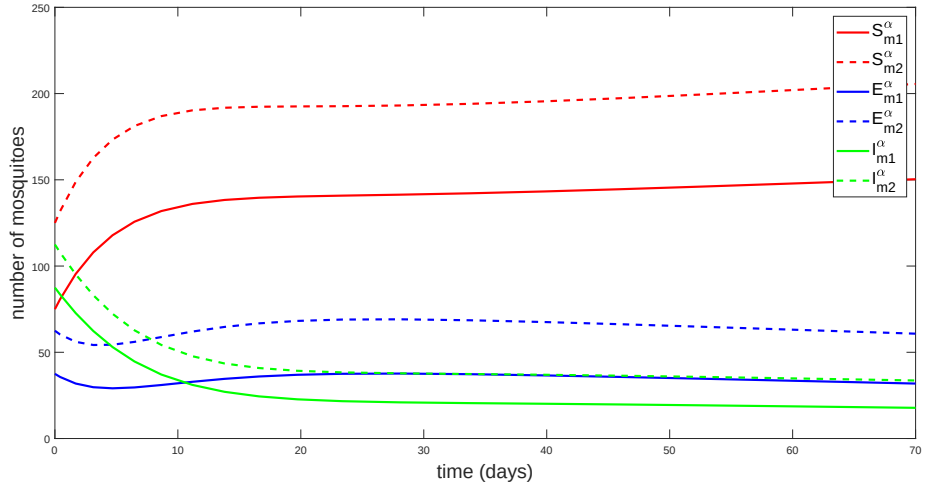

Figure 3: Endemic trends of mosquito populations at level $\alpha = 0.5$ with parameter values given in Table 1 and initial conditions given in section 4.1. The corresponding fuzzy basic reproduction number and the estimated endemic state for mosquito populations are $\mathcal{R}_0^{\alpha} = 3.057$ and $(S^{\alpha}_{m1}, S^{\alpha}_{m2}, E^{\alpha}_{m1}, E^{\alpha}_{m2}, I^{\alpha}_{m1}, I^{\alpha}_{m2}) = (150, 206, 32, 61, 17, 34)$ respectively.

At the level $\alpha = 0.9$, we have the endemic trends of both human and mosquito populations given in Figures 4 and 5 respectively. At this level, we notice that the lower and upper bounds of each class are very closer. This fact means that as α increases, the lower and upper bounds of the level set α of a corresponding fuzzy variable converge towards a single value.

Figures 6-9 give some disease-free trends of both human and mosquito populations at different levels α. The control will be done in three parameters, the average number n of mosquito bites, the proportion θ of humans who recover and the recovery rate γ of humans. Figures 6 and 7 give the disease-free trends of both human and mosquito populations at the level $\alpha = 0.5$.

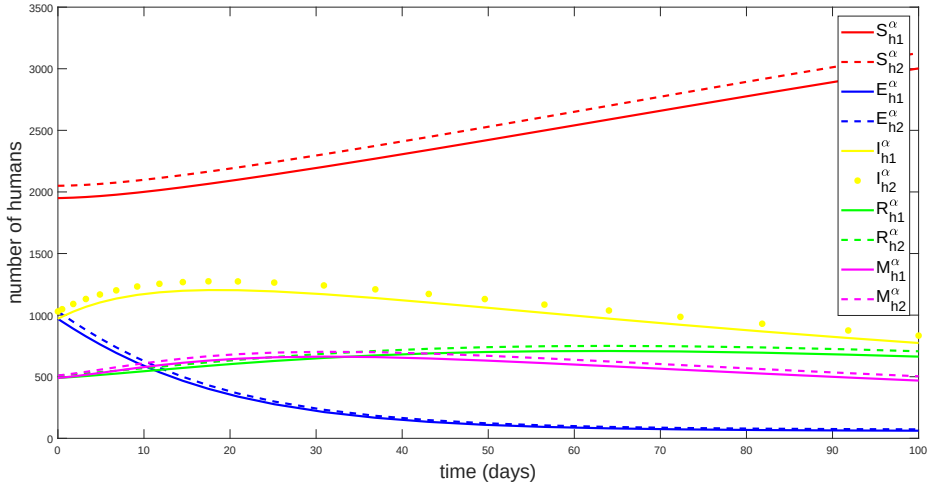

Figure 4: Endemic trends of human populations at level $\alpha = 0.9$ with parameter values given in Table 1 and initial conditions given in section 4.1. The corresponding fuzzy basic reproduction number is $\mathcal{R}_0^{\alpha} = 1.94$. The estimated endemic state for human populations is: $(S_{h1}^{\alpha*}, S_{h2}^{\alpha*}, E_{h1}^{\alpha*}, E_{h2}^{\alpha*}, I_{h1}^{\alpha*}, I_{h2}^{\alpha*}, R_{h1}^{\alpha*}, R_{h2}^{\alpha*}, M_{h1}^{\alpha*}, M_{h2}^{\alpha*}) = (3003, 3128, 73, 74, 774, 834, 663, 707, 469, 504)$.

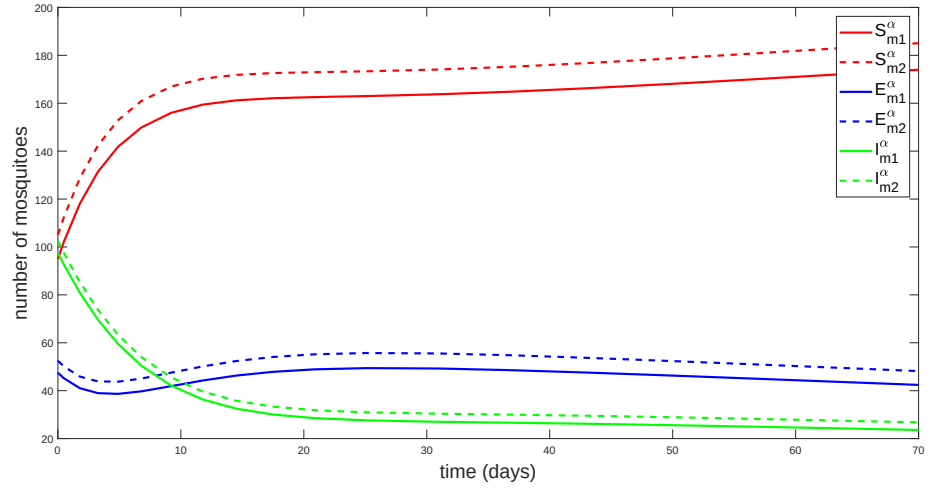

Figure 5: Endemic trends of mosquito populations at level $\alpha = 0.9$ with parameter values given in Table 1 and initial conditions given in section 4.1. The corresponding fuzzy basic reproduction number is $\mathcal{R}_0^{\alpha} = 1.94$. The estimated endemic state for mosquito populations is: $(S_{m1}^{\alpha*}, S_{m2}^{\alpha*}, E_{m1}^{\alpha*}, E_{m2}^{\alpha*}, I_{m1}^{\alpha*}, I_{m2}^{\alpha*}) = (173, 185, 42, 48, 23, 27)$.

At level $\alpha = 0.9$, we have the disease-free trends of both human and mosquito populations given in Figures 8 and 9. As in endemic trends, we can notice that, as the level α increases, the lower and upper bounds of the level set α of a corresponding fuzzy variable converge towards a single value. This fact confirms again the flexibility of fuzzy models comparing with the deterministic one, although we reach to the same control measures as found by Mangongo et al. (2022).

5 Discussion and concluding remarks

In this paper, we analyzed a mathematical model for malaria transmission dynamics with relapse and ignorant infected humans in a fuzzy environment. We assumed that the mosquito

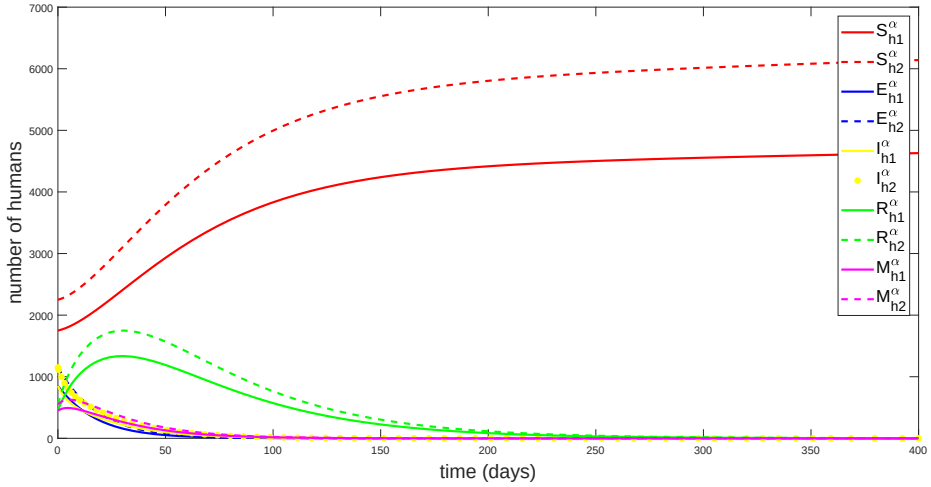

Figure 6: Disease-free trends of human populations at level $\alpha = 0.5$ with parameter values given in Table 1, except $\theta = 0.6$; $\gamma = 0.2$ and $n = 2$. The corresponding fuzzy basic reproduction number is $\mathcal{R}_0^{\alpha} = 0.53$. The estimated disease-free state for human populations is: $(S_{h1}^{\alpha 0}, S_{h2}^{\alpha 0}, E_{h1}^{\alpha 0}, E_{h2}^{\alpha 0}, I_{h1}^{\alpha 0}, I_{h2}^{\alpha 0}, R_{h1}^{\alpha 0}, R_{h2}^{\alpha 0}, M_{h1}^{\alpha 0}, M_{h2}^{\alpha 0}) = (4630, 6141, 0, 0, 0, 0, 0, 0, 0, 0)$.

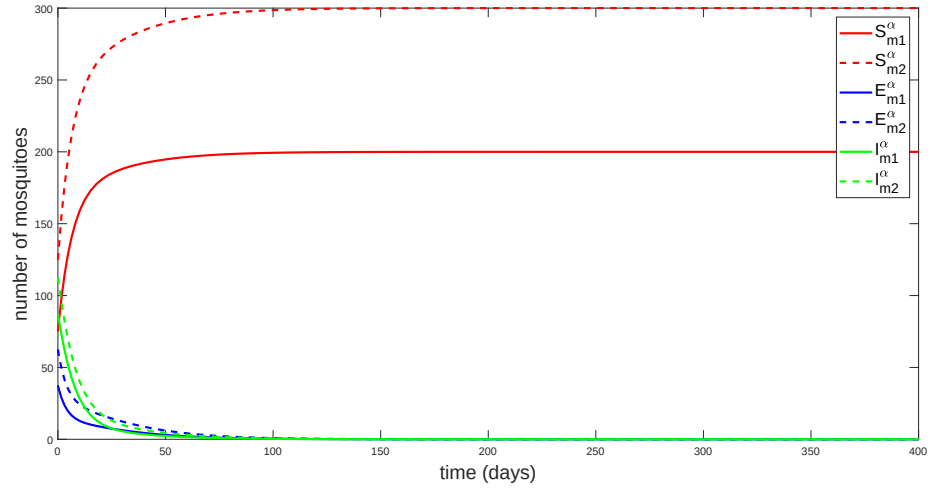

Figure 7: Disease-free trends of mosquito populations at level $\alpha = 0.5$ with parameter values given in Table 1, except $\theta = 0.6$; $\gamma = 0.2$ and $n = 2$. The corresponding fuzzy basic reproduction number is $\mathcal{R}_0^{\alpha} = 0.53$. The estimated disease-free state for mosquito populations is: $(S_{m1}^{\alpha 0}, S_{m2}^{\alpha 0}, E_{m1}^{\alpha 0}, E_{m2}^{\alpha 0}, I_{m1}^{\alpha 0}, I_{m2}^{\alpha 0}) = (200, 300, 0, 0, 0, 0)$.

populations are fuzzy variables. Using level set α, we formulated the deterministic counterpart of the original fuzzy model and we shown that for different levels $\alpha \in]0, 1]$, we can generate several models and doing several analysis. Additionally, we proved that fuzzy models are more general than deterministic one. The equilibrium of the deterministic counterpart of the fuzzy model reveals that in the absence of malaria, the human and mosquito populations belong to the level set α of the fuzzy variables $\tilde{S}_h$ and $\tilde{S}_m$ respectively. We computed and introduced a new definition of the fuzzy basic reproduction number, $\mathcal{R}_0^{\alpha}$ depending on α and we proved that the disease-free equilibrium is globally asymptotically stable in the invariant positive compact set Ω if $\mathcal{R}_0^{\alpha} \leq 1$ for all $\alpha \in]0, 1]$. Otherwise, the disease-free equilibrium is unstable and there exists at least one endemic equilibrium point in the interior of Ω for all $\alpha \in]0, 1]$.

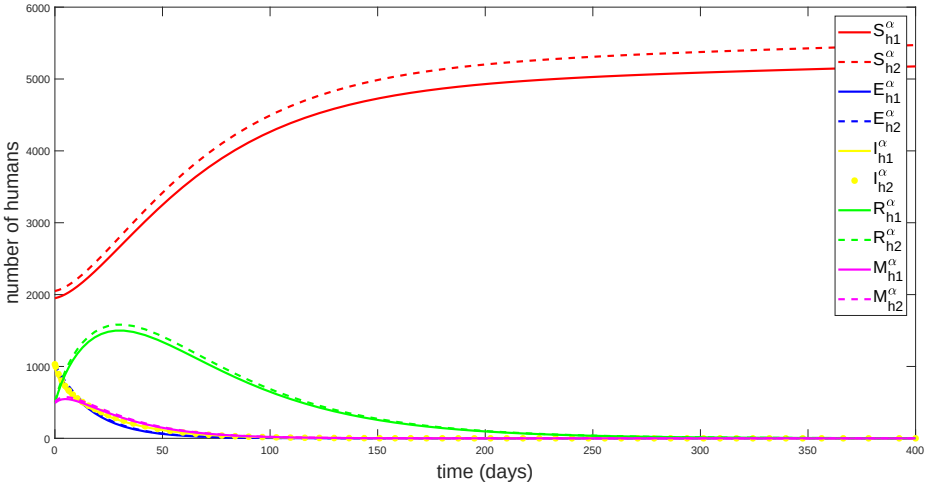

Figure 8: Disease-free trends of human populations at level $\alpha = 0.9$ with parameter values given in Table 1 except $\theta = 0.6$; $\gamma = 0.2$ and $n = 2$. The corresponding fuzzy basic reproduction number is $\mathcal{R}_0^{\alpha} = 0.34$. The estimated disease-free state for human populations is: $(S_{h1}^{\alpha 0}, S_{h2}^{\alpha 0}, E_{h1}^{\alpha 0}, E_{h2}^{\alpha 0}, I_{h1}^{\alpha 0}, I_{h2}^{\alpha 0}, R_{h1}^{\alpha 0}, R_{h2}^{\alpha 0}, M_{h1}^{\alpha 0}, M_{h2}^{\alpha 0}) = (5175, 5472, 0, 0, 0, 0, 0, 0, 0, 0)$.

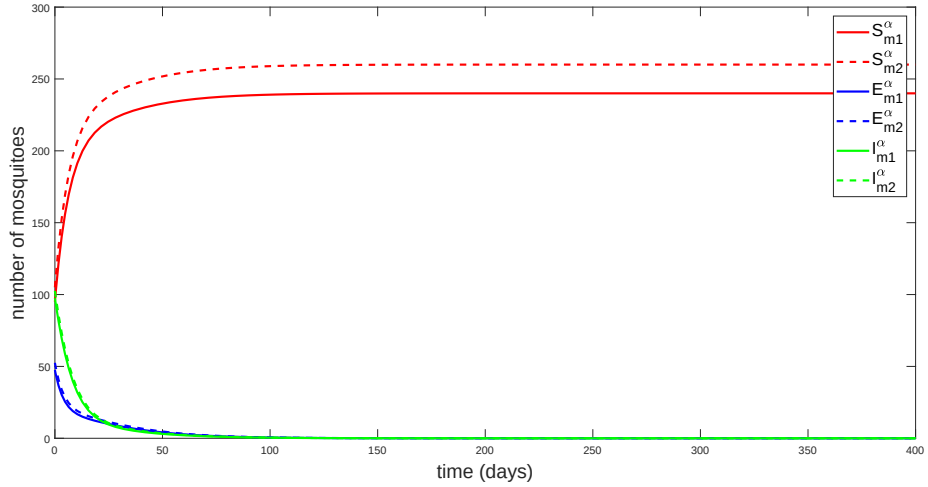

Figure 9: Disease-free trends of mosquito populations at level $\alpha = 0.9$ with parameter values given in Table 1 except $\theta = 0.6$; $\gamma = 0.2$ and $n = 2$. The corresponding fuzzy basic reproduction number is $\mathcal{R}_0^{\alpha} = 0.34$. The estimated disease-free state for mosquito populations is: $(S_{m1}^{\alpha 0}, S_{m2}^{\alpha 0}, E_{m1}^{\alpha 0}, E_{m2}^{\alpha 0}, I_{m1}^{\alpha 0}, I_{m2}^{\alpha 0}) = (240, 260, 0, 0, 0, 0)$.

Figures 6-9 show that as α increases, the fuzzy basic reproduction number $\mathcal{R}_0^{\alpha}$ decreases. Analysis must be done for value of α very closer to one, in order to control malaria. That is, in order to have the fuzzy basic reproduction number less than unity. Table 2 gives the sensitivity analysis and shows that, the level α does not influence the sensitivity indices of parameters. In addition, parameter n has a higher sensitivity index, followed by γ and θ. To control malaria, we have to increase the proportion θ of humans who recover, that is, we must reduce the number of ignorant infected humans in the population. Analysis have been done for different values of α, and taking $\alpha = 0.9$, with parameter values on Table 1, we obtain the endemic trends of both human and mosquito populations. The corresponding fuzzy basic reproduction number is $\mathcal{R}_0^{\alpha} = 1.944$. Taking $\theta = 0.6; \gamma = 0.2; n = 2$ and level $\alpha = 0.9$, we reached the disease-free equilibrium point. Its corresponding fuzzy basic reproduction number is $\mathcal{R}_0^{\alpha} = 0.34$. Analysis must be done for

value of α very closer to unity, because at this level, values has a greatest possibility to occur and we have a smallest value of the fuzzy basic reproduction number ($\mathcal{R}_0^\alpha < 1$).

In this paper, we considered only the mosquito populations as fuzzy variables. We didn't consider other parameters as fuzzy varaibles. For future work, one can consider other parameters as fuzzy variables, such as transmission and recovered rates, to capture very closely the dynamics transmission of malaria. The same concepts of fuzzy sets theory may be applied in the other diseases.

References

Adugna, T.,Getu, E., & Yewhelew, D. (2022). Parous rate and longevity of anophelines mosquitoes in bure district Northwestern Ethiopia. *PLOS ONE*, `https://doi.org/10.1371/journal.pone.0263295`.

Alamin, A. Mondal, S.P., Biswas, K., & Alam, S. (2020). Analysis of discrete system modelling followed by spread of infectious diseases problem in fuzzy environments. *Mathematical Models of Infectious Diseases and Social Issues*, 18.

Allehiany, F., Dayan, F., Al-Harbi, F., Althobaiti, N., Ahmed, N., Rafiq, M., Raza, A. & Elamin, M. (2022). Bio-inspired numerical analysis of covid-19 with fuzzy parameters. *Computers, Materials and Continua*, *72*(5), 3214–3229 DOI: 10.32604/cmc.2022.025811.

Atangan,a A., Qureshi, S. (2020). Mathematical modeling of an autonomous nonlinear dynamical system for malaria transmission using caputo derivative. *Fractional Order Analysis: Theory, Methods and Applications, 22*(5), 252, `https://doi.org/10.1002/9781119654223.ch9`.

Bede, B., Rudas, I.J. & Bencsik, L. (2013). *Mathematics of Fuzzy Sets and Fuzzy Logic*. Studies in Fuzziness and Soft Computing.

Bhuju, G., Phaijoo, G. & Gurung D. (2020). Fuzzy approach analyzing seir-sei dengue dynamics. *Hindawi, BioMed Research International*, Article ID 1508613 `https://doi.org/10.1155/2020/1508613`

Buckley, J.J., Feuring T. (2000). Fuzzy differential equations. *Fuzzy Sets and Systems*, 110, 43-54.

Caraballo, T., Han, X. (2016). *Applied nonautonomous and random dynamical systems: Applied dynamical systems*. Springer.

Dayan, F., Ahmed, N., Rafiq, M., Akgul, A., Raza, A., Ahmad, M.O. & Jarad F. (2022a). Construction and numerical analysis of a fuzzy non-standard computational method for the solution of an seiqr model of covid-19 dynamics. *AIMS Mathematics,*, 7, 8449–8470, DOI: 10.3934/math.2022471.

Dayan, F., Rafiq, M., Ahmed, N., Baleamu, D., Raza, A., Ahmad, M.O. & Iqbal, M. (2022b). Desing and numerical analysis of fuzzy nonstandard computational methods for the solution of rumor based fuzzy epidemic model. *Physica A: Statistical Mechanics and its Applications,* 600, 127542 `https://doi.org/10.1016/j.physa.2022.127542`

Dayan, F., Rafiq, M., Ahmed, N., Raza, A. & Ahmad, M.O. (2022c). A dynamical study of a fuzzy epidemic model of mosquito-borne disease. *Computers in Biology and Medicine,* 105673 `https://doi.org/10.1016/j.compbiomed.2022.105673`

De Barros, L., Tonelli, P., & Bassanezi, R. (2000). Fuzzy modelling in population dynamics. *Ecological modelling*, 128, 27–33.

De Barros, L., Leite, M.F. & Bassanezi, R. (2003). The si epidemiological models with a fuzzy transmission parameter. *An International Journal Computers and Mathematics with Applications*, 45, 1619–1628.

Djidjou-Demasse, R., Abiodun, G.J., Adeola, A.M. & Botai, J.O. (2020). Development and analysis of a malaria transmission mathematical model with seasonal mosquitoe life-history traits. *Studies in Applied Mathematics*, 144, 389-411.

Fard, O.S. (2009). An iterative scheme for the solution of generalized system of linear fuzzy differential equations. *World Applied Sciences Journal*, 7, 1597–1604.

Herdicho, F.F., Chukwu, W., & Tasman, H. (2021). An optimal control of malaria transmission model with mosquito seasonal factor. *Results in Physics*, 25, 104238. https://doi.org/10.1016/j.rinp.2021.104238.

Gazi, K.H., Mondal, S.P., Chatterjee, B., Ghorui, N., Ghosh, A. & De, D. (2023). A new synergitic strategy for ranking restaurant locations: A decision-making approach based on the hexagonal fuzzy numbers. *RAIRO Operations Research*, 57, 571–608 https://doi.org/10.1051/ro/2023025.

Keno, T.D., Obsu, L.L., & Makinde, O.D. (2022). Modeling and optimal control analysis of malaria epidemic in the presence of temperature variability. *Asian-European Journal of Mathematics*, *15*(1).

LaSalle, J.P. (1976). The stability of dynamical systems. *Siam*, 5.

Mangongo, Y.T., Bukweli, J.D.K., & Kampempe, J.D.B. (2021a). Fuzzy global stability analysis of the dynamics of malaria with fuzzy transmission and recovery rates. *American Journal of Operations Research*, 11, 257–282 https://doi.org/10.4236/ajor.2021.116017,

Mangongo, Y.T., Kampempe, J.D.B., & Luhandjula, M.K. (2021). A kaleodoscopic view of fuzzy stochastic optimization. *American Journal of Operations Research*, 11, 283–308, https://doi.org/10.4236/ajor.2021.116018,

Mangongo, Y.T., Bukweli, J.D.K., Kampempe, J.D.B., Mabela, R.M. & Munganga, J.M.W. (2022). Stability and global sensitivity analysis of the transmission dynamics of malaria with relapse and ignorant infected humans. *Physica Scripta, 97*(2), 1–22 https://doi.org/10.1088/1402-4896/ac4862.

Momena, A.F., Mandal, S., Gazi, K.H., Giri, B.C. & Mondal, S.P. (2023). Prediagnosis of disease based on symptoms by generalized dual hesitant hexagonal fuzzy multi-criteria decision-making techniques. *Systems, 11*(5), 231, https://doi.org/10.3390/systems11050231.

Mondal, P.K., Jana, S., Haldar, P. & Kar, T.K. (2015). Dynamical behavior of an epidemic model in a fuzzy transmission. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, *23*(5), 651–665.

Ndamuzi, E., Gahungu, P. (2021). Mathematical modeling of malaria transmission dynamics: case of burundi. *Journal of Applied Mathematics and Physics, 9*(10), 2447–2460 hhttps://doi.org/10.4236/jamp.2021.910156.

Ndoen, E., Wild, C., Dale, P., Sipe, N. & Dale, M. (2012). Mosquito longevity, vector capacity and malaria incidence in west timor and central java, indonesia. *International Scholarly Research Notices*, Article ID 143863, 5 https://doi.org/10.5402/2012/143863

Olaniyi, S., Okosun, K., Adesanya, S. & Lebelo, R. (2020). Modelling malaria dynamics with partial immunity and protected travellers: optimal control and costeffectiveness analysis. *Journal of Biological Dynamics*, 14, 90-115.

Olaniyi, S., Mukamuri, M. Okosun, K. & Adepoju, O. (2022). Mathematical analysis of a social hierarchy-structured model for malaria transmission dynamics. *Results in Physics,* 34, 1–13. https://doi.org/10.1016/j.rinp.2021.104991,

O.M.S. (2022). Estimation sanitaires mondiales de l'oms. note: Classification des pays en fonction de leur revenu établie par la banque mondiale. https//www.who.int/fr/news-room/fact-sheets/detail/the-top-10-causes-of-death,consult\unhbox\voidb@x\bgroup\let\unhbox\voidb@x\setbox\@tempboxa\hbox{e\global\mathchardef\accent@spacefactor\spacefactor}\accent19e\egroup\spacefactor\accent@spacefactorle16/08/2022.

Park, S., Nixon, C.E., Miller, O. Choi, N.-K., Kurtis, J.D., Friedman, J.F. & Michelow, I.C. (2020). Impact of malaria in pregnancy on risk of malaria in young children: Systematic review and meta-analysis. *The Journal of Infectious Diseases*, *222*(4), 538-550.

PNLP. (2018). Evaluation de l'impact des interventions de lutte contre le paludisme sur la mortalité toutes causes confondues chez les enfants de moins de cinq ans en république démocratique du congo de 2005 à 2015. *Programme National de Lutte contre le Paludisme*, Mars 2018.

Seikkala, S. (1987). On the fuzzy initial value problem. *Fuzzy Sets and Systems*, 24, 319–330.

Shuai, Z., van den Driessche, P. (2013). Global stability of infectious disease models using lyapunov functions. *Society for Industrial and Applied Mathematics*, *73*(4), 1513–1532.

Singh, P., Gor, B., Gazi, K.H., Mukherjee, S., Mahata, A. & Mondal, S.P. (2023). Analysis and interpretation of malaria disease model in crisp and fuzzy environment. *Results in Control and Optimization,* 12, 100257 https://doi.org/10.1016/j.rico.2023.100257.

Van den Driessche, P., Watmough, J. (2002). Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Mathematical Biosciences*, 180, 29–48.

Verma, R., Tiwari, S. & Upadhyay, R.K. (2018). Dynamical behaviors of fuzzy sir epidemic model. *Advances in Intelligent Systems and Computing*, 482–492.

W.H.O. World malaria report 2021. Geneve: WHO, https://www.who.int/news-room/fact-sheets/detail/malaria,checkout02/05/2023.

W.H.O. (2010). Basic malaria microscopy-2nd edition. Part 1: Learner's guide. ISBN 978 92 4 154782 6, Geneva.

Wiggins, S., Golubitsky, N. (1990). *Introduction to applied nonlinear dynamical systems and chaos.* Springer.

Zadeh, L.A. (1965). *Fuzzy sets.* University of California, Berkeley, California.

P. Zongo. (2009). *Modélisation mathématique de la dynamique de transmission du paludisme.* PhD thesis, Université de Ouagadougou, 2009.